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Flag manifolds, symmetric $\mathfrak{t}$-triples and Einstein metrics

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ABSTRACT

Let G be a compact connected simple Lie group and let $M = G^{\mathbb{C}}/P = G/K$ be a generalized flag manifold. In this article we focus on an important invariant of G/K, the so-called $\mathfrak{t}$-root system $R_{\mathfrak{t}}$, and we introduce the notion of symmetric $\mathfrak{t}$-triples, that is triples of $\mathfrak{t}$-roots $\xi, \zeta, \eta \in R_{\mathfrak{t}}$ such that $\xi + \eta + \zeta = 0$. We describe their properties and we present an interesting application on the structure constants of G/K, quantities which are straightforward related to the construction of the homogeneous Einstein metric on G/K. We classify symmetric $\mathfrak{t}$-triples for generalized flag manifolds G/K with second Betti number $b_2(G/K) = 1$, and next we treat the case of full flag manifolds G/T, with $b_2(G/T) = \ell = \operatorname{rk} G$, where T is a maximal torus of G. In the last section we construct the homogeneous Einstein equation on flag manifolds G/K with five isotropy summands, determined by the simple Lie group $G = \mathrm{SO}(7)$. By solving the corresponding algebraic system we classify all $\mathrm{SO}(7)$-invariant (non-isometric) Einstein metrics, and these are the very first results towards the classification of homogeneous Einstein metrics on flag manifolds with five isotropy summands.

0. Introduction

Let G be a compact, connected, simple Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. We will denote by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G, by $\varphi(\cdot,\cdot)$ the Killing form of $\mathfrak{g}$, and by $(\cdot,\cdot) = -\varphi(\cdot,\cdot)$ the induced $\mathrm{Ad}(G)$-invariant inner product. Recall that a generalized flag manifold is a complex homogeneous space of the form $M = G^{\mathbb{C}}/P$ where $G^{\mathbb{C}}$ is the unique simply connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and P is a parabolic subgroup of $G^{\mathbb{C}}$. M is diffeomorphic to the coset $G/K = G/C(S)$, where $C(S)$ is the centralizer of a torus $S \subset G$, and thus $M = \mathrm{Ad}(G)w = \{\mathrm{Ad}(g)w\colon\ g \in G\}$ is an adjoint orbit of an element $w \in \mathfrak{g}$. If S is a maximal torus in G, say T, then $C(T) = T$ and we get the full flag manifold $M = G/T$. In this case we have the diffeomorphism $M = G^{\mathbb{C}}/B \cong G/T$, where B is a Borel subgroup of $G^{\mathbb{C}}$. An important invariant which is closely related to the geometry and the structure of a flag manifold $M = G/K$, is the set $R_{\mathfrak{t}}$ of $\mathfrak{t}$-*roots*. These are linear forms obtained by restricting the set R_M of complementary roots of M to the space $\mathfrak{t}$, a real form of the center of the Lie subalgebra $\mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$. They were first introduced by Siebenthal [25], but their current form is due to D.V. Alekseevsky [1,3]. To be more specific, $\mathfrak{t}$-roots are the minimal weights of the irreducible submodules of the isotropy representation of G/K, and thus they have a fundamental role in the Kählerian geometry of G/K. For example, most G-invariant objects on G/K, like as Riemannian metrics, complex structures and Kähler–Einstein metrics, can be expressed in terms of the $\mathfrak{t}$-root system $R_{\mathfrak{t}}$ (cf. [1,3,5]).

In this paper we introduce the notion of *symmetric* $\mathfrak{t}$-*triples*, that is triples of $\mathfrak{t}$-roots $\xi, \zeta, \eta \in R_{\mathfrak{t}}$ with $\xi + \zeta + \eta = 0 \in \mathfrak{t}^*$. Due to correspondence between $\mathfrak{t}$-roots $\xi \in R_{\mathfrak{t}}$ and non-equivalent irreducible submodules $\mathfrak{m}_{\xi}$ of the $\mathrm{Ad}(K)$-module $\mathfrak{m}^{\mathbb{C}}$, symmetric $\mathfrak{t}$-triples are straightforward related with the structure constants c_{ij}^{k} of G/K which are defined as follows: We

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consider an $(\cdot,\cdot)$-orthogonal reductive decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ of $\mathfrak{g}=T_eG$ and as usual we identify the $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ with the tangent space T_oG/K of G/K at the identity $o=eK\in G/K$. Now, we assume that the direct sum decomposition $\mathfrak{m}=T_oG/K=\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_s$ determines an $(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$ into s pairwise inequivalent irreducible $\mathrm{Ad}(K)$-modules (such a decomposition always exists and it is given in terms of $\mathfrak{t}$-roots). We fix an $(\cdot,\cdot)$-orthogonal basis $\{e_\alpha\}$ adapted to the decomposition of $\mathfrak{m}$, that is $e_\alpha\in\mathfrak{m}_i$ for some i, and $\alpha<\beta$ if $i<j$ (with $e_\alpha\in\mathfrak{m}_i$ and $e_\beta\in\mathfrak{m}_j$), and we set $A^\gamma_{\alpha\beta}=([e_\alpha,e_\beta],e_\gamma)$ such that $[e_\alpha,e_\beta]_{\mathfrak{m}}=\sum_\gamma A^\gamma_{\alpha\beta}X_\gamma$, where $[\,,]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component. Then, the structure constants of G/K with respect to the decomposition $\mathfrak{m}=\bigoplus_{i=1}^s\mathfrak{m}_i$, are given by

$$c^k_{ij}:=\begin{bmatrix}k\\ ij\end{bmatrix}:=\sum\left(A^\gamma_{\alpha\beta}\right)^2=\sum\big(([e_\alpha,e_\beta],e_\gamma)\big)^2, \tag{1}$$

where the sum is taken over all indices α,β,γ with $e_\alpha\in\mathfrak{m}_i$, $e_\beta\in\mathfrak{m}_j$, $e_\gamma\in\mathfrak{m}_k$, and $i,j,k\in\{1,\ldots,s\}$ [26]. In Section 1 we will show that symmetric $\mathfrak{t}$-triples in $\mathfrak{t}^*$ are in a bijective correspondence with the non-zero c^k_{ij}. Since these quantities are closely related with the construction of the homogeneous Einstein equation, it turns out that symmetric $\mathfrak{t}$-triples have a key role in the related theory of Einstein metrics.

Recall that a Riemannian manifold (M,g) is called *Einstein* if it has constant Ricci curvature, i.e. $\mathrm{Ric}_g=\lambda\cdot g$, where $\lambda\in\mathbb{R}$ is the so-called *Einstein constant*. The Einstein equation forms a system of non-linear second order PDEs and a good understanding of its solutions in the general case seems far from being attained. It is more manageable when a Lie group G of isometries acts on the manifold M, via various ways. In the homogeneous case and for a G-invariant Riemannian metric, the Einstein equation reduces to a system of algebraic equations which in some cases can been solved explicitly. However, even in this case, general existence or non-existence results are difficult to obtained (cf. [17,26,22,15]). We mention that for a homogeneous Riemannian manifold $(M=G/K,g)$ of a compact connected (semi-)simple Lie group G, the Ricci tensor is expressed in terms of the structure constants c^k_{ij}, the parameters x_i which define the G-invariant metric tensor g with respect to the decomposition $\mathfrak{m}=\bigoplus_{i=1}^s\mathfrak{m}_i$, and the dimensions $d_i=\dim_{\mathbb{R}}\mathfrak{m}_i$ for any $i=1,\ldots,s$. The determination as well as the computation of all the non-zero c^k_{ij}, are usually non-trivial problems towards to the formulation of the equation $\mathrm{Ric}_g=\lambda\cdot g$ on $(M=G/K,g)$, especially when the number of isotropy summands increases.

Recently it has been a lot of progress on homogeneous Einstein metrics on flag manifolds. For example, they have been completely classified for any flag manifold $M=G/K$ (of a compact simple Lie group G) with two [24,6,4], three [20,5,4], or four isotropy summands [7,8,10]. Homogeneous Einstein metrics on full flag manifolds corresponding to classical Lie groups have been also studied by several authors (cf. [5,23,16]). Moreover, in a recent work of the author in collaboration with A. Arvanitoyeorgos and Y. Sakane [11], all G_2-invariant Einstein metrics were obtained on the exceptional full flag manifold G_2/T (a homogeneous space with six isotropy summands). A further study on invariant Einstein metrics on $\mathrm{Sp}(n)$-flag manifolds whose $\mathfrak{t}$-root system is of the same form of the full flag manifold G_2/T, namely of G_2-*type*, was given in [9]. Flag manifolds whose isotropy representation decomposes into more than four isotropy summands are treated also in [14], as well as in [17]. Another alternative approach to homogeneous Einstein metrics, has been recently established by the author in collaboration with S. Anastassiou [4]; in this paper the global behaviour of the normalized Ricci flow on the space of G-invariant metrics for a flag manifold G/K was studied, and invariant Einstein metrics were obtained explicitly as the singularities of this flow, located at infinity. This approach seems to give a better insight on the behaviour of the Einstein equation and as it has been mentioned in [4], it worths further investigation for a better understanding of its benefits (see also [18]).

In this article we classify all invariant Einstein metrics on flag spaces G/K with five isotropy summands, corresponding to the simple Lie group $G=\mathrm{SO}(7)$. There are only two such cosets, defined by the subsets $\Pi_{M_1}=\{\alpha_1,\alpha_3\}\subset\Pi$, and $\Pi_{M_2}=\{\alpha_2,\alpha_3\}\subset\Pi$, respectively, where $\Pi=\{\alpha_1,\alpha_2,\alpha_3\}$ is a system of simple roots for the root system of SO(7). Both of them are given by $M=\mathrm{SO}(7)/\mathrm{U}(1)\times\mathrm{U}(2)\cong\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SU}(2)$, and as we see in Section 3, these flag manifolds are isometric (as real manifolds), and thus we will not distinguish them. Note that if we paint black both the first two simple roots in the Dynkin diagram of SO(7), that is $\Pi_M=\{\alpha_1,\alpha_2\}$, then we obtain also a flag manifold of the form $\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SU}(2)\cong\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SO}(3)$, but it has four isotropy summands (see [7]). We use the theory of symmetric $\mathfrak{t}$-triples and we determine all the non-zero structure constants c^k_{ij} of M, with respect to the decomposition $\mathfrak{m}=\bigoplus_{i=1}^5\mathfrak{m}_i$. For their computation, we use a Kähler–Einstein metric corresponding to an SO(7)-invariant complex structure J, induced by an invariant ordering R^+_M on the set of the complementary roots of M. In this way we write down explicitly the Ricci tensor and thus the algebraic system which determines the homogeneous Einstein equation. We prove the following theorem:

Theorem A. *The flag manifold $M=\mathrm{SO}(7)/\mathrm{U}(1)\times\mathrm{U}(2)\cong\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SU}(2)$, defined by the set $\Pi_{M_1}=\{\alpha_1,\alpha_3\}\subset\Pi$, or the set $\Pi_M=\{\alpha_2,\alpha_3\}\subset\Pi$, admits two pairs of isometric* SO(7)*-invariant Kähler–Einstein metrics. There are also four* SO(7)*-invariant Einstein metrics, which are not Kähler with respect to any invariant complex structure on M. Two of them are isometric, thus M admits three* (*up to isometry*) *non-Kähler–Einstein metrics.*

We have divided the paper into 3 sections. In Section 1 we review the Lie theoretic description of a flag manifold $M=G/K$, we introduce the notion of symmetric $\mathfrak{t}$-triples and we state their relation with structure constants. In Section 2, we focus on flag manifolds $M=G/K$ with $b_2(M)=1$ and we prove a structure theorem related to the associated $\mathfrak{t}$-root system $R_{\mathfrak{t}}$ (cf. Theorem 2.1). Based on this result we obtain the full classification of symmetric $\mathfrak{t}$-triples for such flag manifolds and we present them for any case. Next we extend our study of symmetric $\mathfrak{t}$-triples to full flag manifolds $M = G/T$; recall that here we have $b_2(G/T) = \ell = \operatorname{rk} G = \dim_{\mathbb{R}} T$. In the final Section 3, we give the reader a quick view of the Einstein equation on a homogeneous Riemannian manifold, and next we prove Theorem A.

1. Flag manifolds and symmetric $\mathfrak{t}$-triples

Let $M = G^{\mathbb{C}}/P = G/C(S) = G/K$ be a generalized flag manifold of a compact, connected, simple Lie group, where S is a torus in G. We begin by providing an algebraic description of M. We follow the notation of Introduction, and we denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ the complexifications of the Lie algebras of G and K, respectively. We fix a maximal torus T in G which contains the torus S and we denote by $\mathfrak{h} = T_eT$ its Lie algebra and by $\mathfrak{h}^{\mathbb{C}}$ the corresponding complexification. Since $S \subset T \subset C(S) = K$, it follows that T is a maximal torus also for the isotropy subgroup K and thus for any flag manifold $M = G/K$ it holds $\operatorname{rk} G = \operatorname{rk} K$. Let $R \subset (\mathfrak{h}^{\mathbb{C}})^* \backslash \{0\}$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_{\alpha}^{\mathbb{C}}$ be the associated root space decomposition. We consider vectors $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ which are defined by $\varphi(H, H_\alpha) = \alpha(H)$, for all $H \in \mathfrak{h}^{\mathbb{C}}$, and let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ ($\ell = \dim \mathfrak{h}^{\mathbb{C}}$) be a basis of simple roots for R. We will denote by R^+ the induced ordering. We set $A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$, where $E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$ ($\alpha \in R^+$) is a Weyl basis of $\mathfrak{g}^{\mathbb{C}}$ (i.e. $\varphi(E_\alpha, E_{-\alpha}) = -1$ and $[E_\alpha, E_{-\alpha}] = -H_\alpha$). Then, the real Lie algebra $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$ and it is identified with the fixed point set $\mathfrak{g}^\tau$ of the conjugation $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, defined by $\tau(E_\alpha) = E_{-\alpha}$. Thus it is $\mathfrak{g}^\tau = \mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$.

Because $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, there is a closed subsystem R_K of R such that $\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha^{\mathbb{C}}$. Indeed, we can always find a subset $\Pi_K \subset \Pi$ such that $R_K = R \cap \langle \Pi_K \rangle = \{\beta \in R\colon\ \beta = \sum_{\alpha_i \in \Pi_K} k_i\alpha_i,\ k_i \in \mathbb{Z}\}$, where $\langle \Pi_K \rangle$ is the space of roots generated by Π_K with integer coefficients. The complex Lie algebra $\mathfrak{k}^{\mathbb{C}}$ is a maximal reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$, so we get the decomposition $\mathfrak{k}^{\mathbb{C}} = Z(\mathfrak{k}^{\mathbb{C}}) \oplus \mathfrak{k}_{ss}^{\mathbb{C}}$, where $Z(\mathfrak{k}^{\mathbb{C}})$ is the center of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{k}_{ss}^{\mathbb{C}} = [\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}]$ is its semi-simple part. In particular, it is $\mathfrak{k}_{ss}^{\mathbb{C}} = \mathfrak{h}' \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha^{\mathbb{C}} = \sum_{\alpha \in \Pi_K} \mathbb{C}H_\alpha \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha^{\mathbb{C}}$, where $\mathfrak{h}' = \sum_{\alpha \in \Pi_K} \mathbb{C}H_\alpha \subset \mathfrak{h}^{\mathbb{C}}$ is the Cartan subalgebra of $\mathfrak{k}_{ss}^{\mathbb{C}}$. Thus, R_K is the root system of the semi-simple part $\mathfrak{k}_{ss}^{\mathbb{C}}$ with a basis of simple roots given by Π_K, i.e. $\dim_{\mathbb{C}} \mathfrak{h}' = |\Pi_K|$, where $|\Pi_K|$ denotes the cardinality of Π_K. The real Lie algebra $\mathfrak{k}$ of K (which is a reductive Lie subalgebra of $\mathfrak{g}$), has the form $\mathfrak{k} = \mathfrak{h} \oplus \sum_{\alpha \in R_K^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$, where $R_K^+ = R^+ \cap \langle \Pi_K \rangle$ is the ordering in R_K induced by R^+. The center $\mathfrak{s} = Z(\mathfrak{k})$ of $\mathfrak{k}$ is given by $\mathfrak{s} = i\mathfrak{t}$, where the subspace $\mathfrak{t}$ is defined by (cf. [1,5]):

$$\mathfrak{t} = Z\big(\mathfrak{k}^{\mathbb{C}}\big) \cap i\mathfrak{h} = \big\{X \in \mathfrak{h}\colon\ \phi(X) = 0 \text{ for all } \phi \in R_K\big\}.$$

To be more specific, $\mathfrak{t}$ is a real form of the center $Z(\mathfrak{k}^{\mathbb{C}})$, i.e. $\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{k}_{ss}^{\mathbb{C}}$. Because $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{k}$ we get the orthogonal splitting $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{t}'$, where $\mathfrak{t}' = \operatorname{span}\{iH_\beta\colon\ \beta \in \Pi_K\}$. Thus the Cartan subalgebra $\mathfrak{h}'$ is given by $\mathfrak{h}' = (\mathfrak{t}')^{\mathbb{C}}$, which means that $\dim_{\mathbb{R}} \mathfrak{t}' = \dim_{\mathbb{C}} \mathfrak{h}' = |\Pi_K|$. Hence $\dim_{\mathbb{R}} \mathfrak{t} = \ell - |\Pi_K|$, where $\ell = \operatorname{rk} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{h} = \dim_{\mathbb{R}} T$. By [12, p. 507], it is $H^2(M;\mathbb{R}) = H^1(K;\mathbb{R}) = \mathfrak{t}$, hence the second Betti number of $M = G/K$ is equal to $b_2(M) = \dim_{\mathbb{R}} H^2(M;\mathbb{R}) = \dim_{\mathbb{R}} \mathfrak{t}$ and it is obtained directly from the painted Dynkin diagram (see below).

We set $\Pi_M := \Pi \backslash \Pi_K$, $R_M := R \backslash R_K$ and $R_M^+ := R^+ \backslash R_K^+$, such that $\Pi = \Pi_K \sqcup \Pi_M$, $R = R_K \sqcup R_M$, and $R^+ = R_K^+ \sqcup R_M^+$, respectively. In a sense, these sets of roots characterize the flag manifold $M = G/K$. Roots in R_M are called *complementary roots* and have a key role in theory which we will describe below. We recall that an *invariant ordering* R_M^+ in R_M is the choice of a subset $R_M^+ \subset R_M$ which satisfies the splitting $R = R_K \sqcup R_M^+ \sqcup R_M^-$, where $R_M^- = \{-\alpha\colon\ \alpha \in R_M^+\} = -R_M^+$, such that:

(i) $\alpha, \beta \in R_M^+$, $\alpha + \beta \in R_M \Rightarrow \alpha + \beta \in R_M^+$,
(ii) $\alpha \in R_M^+$, $\beta \in R_K^+$, $\alpha + \beta \in R \Rightarrow \alpha + \beta \in R_M^+$.

We say that $\alpha > \beta$ if and only if $\alpha - \beta \in R_M^+$. Invariant orderings $R_M^+ \subset R_M$ are very useful. For example, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $(\cdot,\cdot)$-orthogonal reductive decomposition. Then we see that the $\operatorname{Ad}(K)$-module $\mathfrak{m} = T_oM$ has the form $\mathfrak{m} = T_o(G/K) = \sum_{\alpha \in R_M^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$. Moreover, for the complexified version $\mathfrak{m}^{\mathbb{C}}$ we get the expression $\mathfrak{m}^{\mathbb{C}} = T_o(G/K)^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathbb{C}E_\alpha$. Now, all information contained in the splitting $\Pi = \Pi_K \sqcup \Pi_M$ can be presented graphically by the painted Dynkin diagram of $M = G/K$.

Definition 1.1. Let $\Gamma = \Gamma(\Pi)$ be the Dynkin diagram of the fundamental system Π. By painting in black the nodes of Γ corresponding to $\Pi_M := \Pi \backslash \Pi_K$, we obtain the painted Dynkin diagram (PDD) of the flag manifold G/K. In this diagram the subsystem Π_K is determined by the subdiagram of white roots and each black node gives rise to one U(1)-component, which their totality forms the center of K.

From now on we fix a basis $\Pi = \{\alpha_1, \ldots, \alpha_r, \phi_1, \ldots, \phi_k\}$ of R, such that $r + k = \ell = \operatorname{rk} \mathfrak{g}^{\mathbb{C}}$ and we assume that $\Pi_K = \{\phi_1, \ldots, \phi_k\}$ is a basis of the root system R_K of K. We set $\Pi_M = \Pi \backslash \Pi_K = \{\alpha_1, \ldots, \alpha_r\}$. Let $\Lambda_1, \ldots, \Lambda_r$ be the fundamental weights corresponding to the simple roots of Π_M, i.e. the linear forms defined by the relations $2\varphi(\Lambda_i, \alpha_j)/\varphi(\alpha_j, \alpha_j) = \delta_{ij}$ and $\varphi(\Lambda_j, \phi_i) = 0$, where $\varphi(\alpha, \beta)$ denotes the inner product on $(\mathfrak{h}^{\mathbb{C}})^*$ given by $\varphi(\alpha, \beta) = \varphi(H_\alpha, H_\beta)$, for all $\alpha, \beta \in (\mathfrak{h}^{\mathbb{C}})^*$. It is well known that $\{\Lambda_i\colon\ 1 \leqslant i \leqslant r\}$ is a basis of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$, thus $\mathfrak{t}^* = \sum_{i=1}^r \mathbb{R}\Lambda_i$ and $\dim \mathfrak{t}^* = \dim \mathfrak{t} = r$ [3].

Consider the restriction map $\kappa : \mathfrak{h}^* \to \mathfrak{t}^*$ defined by $\kappa(\alpha) = \alpha|_{\mathfrak{t}}$, and set $R_{\mathfrak{t}} = \kappa(R) = \kappa(R_M)$ and $\kappa(R_K) = 0 \in \mathfrak{t}^*$. The elements of $R_{\mathfrak{t}}$ are called $\mathfrak{t}$-*roots*. For an invariant ordering $R_M^+ = R^+ \backslash R_K^+$ in R_M, we set $R_{\mathfrak{t}}^+ = \kappa(R_M^+)$, $R_{\mathfrak{t}}^- = -R_{\mathfrak{t}}^+ = \{-\xi\colon \xi \in R_{\mathfrak{t}}^+\}$ and we get the splitting $R_{\mathfrak{t}} = R_{\mathfrak{t}}^+ \sqcup R_{\mathfrak{t}}^-$, which defines an ordering in $R_{\mathfrak{t}}$ (cf. [7]); $\mathfrak{t}$-toots $\xi \in R_{\mathfrak{t}}^+$ (resp. $\xi \in R_{\mathfrak{t}}^-$) are called *positive* (resp. *negative*). A $\mathfrak{t}$-root is called *simple* if it is not a sum of two positive $\mathfrak{t}$-roots. The set $\Pi_{\mathfrak{t}}$ of all simple $\mathfrak{t}$-roots, the so-called $\mathfrak{t}$-*basis*, is a basis of $\mathfrak{t}^*$ in the sense that any $\mathfrak{t}$-root can be written as a linear combination of its elements with integer coefficients of the same sign. According to [3,7], a $\mathfrak{t}$-basis $\Pi_{\mathfrak{t}}$ is obtained by restricting the roots of Π_M to $\mathfrak{t}$, that is $\Pi_{\mathfrak{t}} = \{\bar{\alpha}_i = \alpha_i|_{\mathfrak{t}}\colon \alpha_i \in \Pi_M\}$. This allows us to set up a useful method to obtain explicitly the set $R_{\mathfrak{t}}$ (see [2,7]).

Proposition 1.2. *(See [25,1,3].) There exists a bijective correspondence between $\mathfrak{t}$-roots and inequivalent complex irreducible $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-submodules $\mathfrak{m}_\xi$ of $\mathfrak{m}^{\mathbb{C}}$, given by*

$$R_{\mathfrak{t}} \ni \xi \quad \leftrightarrow \quad \mathfrak{m}_\xi = \sum_{\alpha \in R_M\colon \kappa(\alpha)=\xi} \mathbb{C}E_\alpha = \sum_{\alpha \in R_M\colon \kappa(\alpha)=\xi} \mathfrak{g}_\alpha^{\mathbb{C}}. \tag{2}$$

Thus $\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi$. Consequently, for the real $\mathrm{ad}(\mathfrak{k})$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbb{C}})^\tau$ we have the following decomposition into real pairwise inequivalent irreducible $\mathrm{ad}(\mathfrak{k})$-submodules: $\mathfrak{m} = \sum_{\xi \in R_{\mathfrak{t}}^+ = \kappa(R_M^+)} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau$.

In order to study the properties of the decomposition $\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi$, the following lemma due to M. Graev is very crucial (for a proof see for example [2]).

Lemma 1.3 *(Graev). Let ξ, η, ζ be $\mathfrak{t}$-roots such that $\xi + \eta + \zeta = 0$. Then, there exist roots $\alpha, \beta, \gamma \in R$ with $\kappa(\alpha) = \xi$, $\kappa(\beta) = \eta$, $\kappa(\gamma) = \zeta$, and such that $\alpha + \beta + \gamma = 0$.*

By using Graev's lemma and the properties of the root spaces $\mathfrak{g}_\alpha^{\mathbb{C}}$ (cf. [19, p. 168]), we obtain that

Corollary 1.4. *Let $M = G/K$ be a flag manifold and let $R_{\mathfrak{t}}$ be its $\mathfrak{t}$-root system. Then:*

(1) *If $\xi, \eta \in R_{\mathfrak{t}}$ are such that $\xi + \eta \neq 0$, then $(\mathfrak{m}_\xi, \mathfrak{m}_\eta) = 0$.*
(2) *If $\xi, \eta \in R_{\mathfrak{t}}$ such that $\xi + \eta \neq 0$, $\xi + \eta \in R_{\mathfrak{t}}$, then $[\mathfrak{m}_\xi, \mathfrak{m}_\eta] \neq 0$. In particular, it is $[\mathfrak{m}_\xi, \mathfrak{m}_\eta] = \mathfrak{m}_{\xi+\eta}$.*

Proof. (1) Since by assumption it is $\xi + \eta \neq 0$ there is a $\mathfrak{t}$-root, say ζ, such that $\xi + \eta + (-\zeta) = 0$. Then, by Lemma 1.3, we can find complementary roots $\alpha, \beta, \gamma \in R_M$ with $\kappa(\alpha) = \xi$, $\kappa(\beta) = \eta$, $\kappa(-\gamma) = -\zeta$ and such that $\alpha + \beta + (-\gamma) = 0$. But then $\alpha + \beta = \gamma \neq 0$ and thus for the associated root spaces we get $(\mathfrak{g}_\alpha^{\mathbb{C}}, \mathfrak{g}_\beta^{\mathbb{C}}) = 0$. Hence, in view of (2) it follows that $(\mathfrak{m}_\xi, \mathfrak{m}_\eta) = 0$.

(2) We use again Lemma 1.3 to find roots $\alpha, \beta, \gamma \in R_M$ with $\alpha + \beta = \gamma \neq 0$. It means that $\alpha + \beta \in R_M \subset R$ and hence $[\mathfrak{g}_\alpha^{\mathbb{C}}, \mathfrak{g}_\beta^{\mathbb{C}}] = \mathfrak{g}_{\alpha+\beta}^{\mathbb{C}} = \mathfrak{g}_\gamma^{\mathbb{C}} \neq 0$. It is also $R_{\mathfrak{t}}^+ \ni \kappa(\alpha + \beta) = \kappa(\alpha) + \kappa(\beta) = \xi + \eta = \zeta$, i.e. $\kappa(\alpha + \beta) \neq 0$, and thus $[\mathfrak{m}_\xi, \mathfrak{m}_\eta] = \mathfrak{m}_\zeta \neq 0$. Since $\mathfrak{m}_\zeta = \sum_{\substack{\gamma \in R_M \\ \kappa(\gamma)=\zeta}} \mathfrak{g}_\gamma^{\mathbb{C}} = \sum_{\substack{(\alpha+\beta) \in R_M \\ \kappa(\alpha+\beta)=\xi+\eta}} \mathfrak{g}_{\alpha+\beta}^{\mathbb{C}}$, it follows that $[\mathfrak{m}_\xi, \mathfrak{m}_\eta] = \mathfrak{m}_{\xi+\eta}$. $\square$

Let us now focus on the real $\mathrm{Ad}(K)$-module $\mathfrak{m} = \sum_{\xi \in R_{\mathfrak{t}}^+ = \kappa(R_M^+)} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau$. For simplicity we assume that $R_{\mathfrak{t}}^+ = \{\xi_1, \ldots, \xi_s\}$. Then, we have the decomposition $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$ where each real irreducible $\mathrm{ad}(\mathfrak{k})$-submodule $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau$ $(1 \leqslant i \leqslant s)$ corresponding to the positive $\mathfrak{t}$-root ξ_i, is given by

$$\mathfrak{m}_i = \sum_{\alpha \in R_M^+\colon \kappa(\alpha)=\xi_i} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha). \tag{3}$$

Eq. (3) shows that an $(\cdot,\cdot)$-orthogonal basis of the component $\mathfrak{m}_i$ consists of the vectors $\{A_\alpha = (E_\alpha + E_{-\alpha}),\ B_\alpha = i(E_\alpha - E_{-\alpha})\}$, where the complementary roots $\alpha \in R_M^+$ are such that $\kappa(\alpha) = \xi_i$, for any $1 \leqslant i \leqslant s$. For simplicity, we denote such a basis by $\{v_i\} = \{A_\alpha, B_\alpha\colon\ \alpha \in R_M^+,\ \kappa(\alpha) = \xi_i \in R_{\mathfrak{t}}^+\}$, for any $1 \leqslant i \leqslant s$. Thus, $\dim_{\mathbb{R}} \mathfrak{m}_i = |\{\pm\alpha \in R_M\colon\ \kappa(\pm\alpha) = \pm\xi_i\}|$.

Remark 1.5. By Corollary 1.4 (2), it follows that for any $\alpha, \beta \in R_M^+$ with $\alpha + \beta \in R$, it is always $[e_\alpha, e_\beta] = e_{\alpha+\beta} \in \{v_{i+j}\}$, where $e_\alpha \in \{v_i\}$, $e_\beta \in \{v_j\}$, and $\{v_{i+j}\} = \{A_{\alpha+\beta}, B_{\alpha+\beta}\colon\ \alpha + \beta \in R_M^+,\ \kappa(\alpha + \beta) = \xi_i + \xi_j \in R_{\mathfrak{t}}^+\}$.

Definition 1.6. A *symmetric* $\mathfrak{t}$-*triple* in $\mathfrak{t}^*$ is a triple $\Xi = (\xi, \eta, \zeta)$ of $\mathfrak{t}$-roots $\xi, \eta, \zeta \in R_{\mathfrak{t}}$ such that $\xi + \eta + \zeta = 0$.

We mention that a symmetric $\mathfrak{t}$-triple $\Xi = (\xi, \eta, \zeta)$ always remains invariant under a permutation of its components ξ, η and ζ, i.e. under the action of the symmetric group $\mathcal{S}_3$. Moreover, because in $R_{\mathfrak{t}}$ we have a polarization $R_{\mathfrak{t}} = R_{\mathfrak{t}}^+ \sqcup R_{\mathfrak{t}}^-$, we associate to any symmetric $\mathfrak{t}$-triple $\Xi = (\xi, \eta, \zeta)$ a negative one, given by $-\Xi = (-\xi, -\eta, -\zeta) = -(\xi, \eta, \zeta)$. Note also that given a symmetric $\mathfrak{t}$-triple $\Xi = (\xi, \eta, \zeta)$ and an integer $\lambda \in \mathbb{Z}^*$, $\lambda \neq \pm 1$, such that $\lambda\xi, \lambda\eta, \lambda\zeta \in R_{\mathfrak{t}}$, then $\lambda\xi + \lambda\eta + \lambda\zeta = \lambda(\xi + \eta + \zeta) = 0$. Therefore we can define a new symmetric $\mathfrak{t}$-triple, say $\lambda\Xi$, given by $\lambda\Xi = \lambda(\xi, \eta, \zeta) = (\lambda\xi, \lambda\eta, \lambda\zeta)$.

Definition 1.7. Two symmetric $\mathfrak{t}$-triples $\Xi = (\xi, \eta, \zeta)$, $\Xi' = (\xi', \eta', \zeta')$ in $R_{\mathfrak{t}}$ are called equivalent if and only if $\Xi = \pm\Xi'$.

Lemma 1.8. *Let $M = G/K$ be a generalized flag manifold and let $R_{\mathfrak{t}}$ be the associated $\mathfrak{t}$-root system. Given a symmetric $\mathfrak{t}$-triple $\Xi = (\xi, \eta, \zeta)$, the following are true*:

(1) *Ξ cannot contain only positive, or only negative $\mathfrak{t}$-roots, i.e. it cannot be $\xi, \eta, \zeta \in R_{\mathfrak{t}}^+$, or $\xi, \eta, \zeta \in R_{\mathfrak{t}}^-$.*
(2) *Ξ cannot contain simultaneously a $\mathfrak{t}$-root and its negative.*

Proof. (1) We will give a proof for positive $\mathfrak{t}$-roots, and for negative $\mathfrak{t}$-roots it is similar. Let $\Xi = (\xi, \eta, \zeta)$ be a symmetric $\mathfrak{t}$-triple such that $\xi, \eta, \zeta \in R_{\mathfrak{t}}^+$. By Lemma 1.3, there exist roots $\alpha, \beta, \gamma \in R_M$ with $\kappa(\alpha) = \xi$, $\kappa(\beta) = \eta$ and $\kappa(\gamma) = \zeta$, such that $\alpha + \beta + \gamma = 0$. But since $\xi, \eta, \zeta \in R_{\mathfrak{t}}^+$ and $R_{\mathfrak{t}}^+ = \kappa(R_M^+)$ it must be $\alpha, \beta, \gamma \in R_M^+$. Since $\alpha + \beta + \gamma = 0$ we get that $\alpha + \beta = -\gamma \in R_M^-$, a contradiction.

(2) Let $\Xi = (\xi, \eta, \zeta)$ be a symmetric $\mathfrak{t}$-triple such that $\eta = -\xi$. Then, because $\kappa(\pm\alpha) = \pm\xi$ for some $\alpha \in R_M$, it must be $\beta = -\alpha$ in the relation $\alpha + \beta + \gamma = 0$, where the complementary roots $\beta, \gamma \in R_M$ are such that $\kappa(\beta) = \eta$ and $\kappa(\gamma) = \zeta$. Thus we conclude that $\gamma = 0$, which is a contradiction since $\gamma \in R_M$. Similarly all possible combinations are treated. $\square$

Corollary 1.9. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G and let $R_{\mathfrak{t}}$ be the associated $\mathfrak{t}$-root system. Assume that $\mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i$ is an $(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{ad}(\mathfrak{k})$-modules, and let $\xi_i, \xi_j, \xi_k \in R_{\mathfrak{t}}$ be the $\mathfrak{t}$-roots associated to the components $\mathfrak{m}_i$, $\mathfrak{m}_j$ and $\mathfrak{m}_k$, respectively. Then, $c_{ij}^k = \left[{k \atop ij}\right] \neq 0$, if and only if (ξ_i, ξ_j, ξ_k) is a symmetric $\mathfrak{t}$-triple, i.e. $\xi_i + \xi_j + \xi_k = 0$.*

Proof. If $c_{ij}^k \neq 0$, for some indices $i, j, k \in \{1, \dots, s\}$, then $\varphi([\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k) \neq 0$. We will prove that $\xi_i + \xi_j + \xi_k = 0$. For any $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau$ we choose an orthogonal basis $\{u_i\} = \{A_\alpha, B_\alpha\colon\ \alpha \in R_M^+,\ \kappa(\alpha) = \xi_i \in R_{\mathfrak{t}}^+\}$. Then, from Remark 1.5 we know that for any $e_\alpha \in \{v_i\}$ and $e_\beta \in \{v_j\}$, it is $[e_\alpha, e_\beta] = e_{\alpha+\beta} \in \{v_{i+j}\}$, unless $\alpha + \beta = 0$. Moreover, by Corollary 1.4 (1), it follows that for some $e_\gamma \in \{v_k\}$ it is $\phi([e_\alpha, e_\beta], e_\gamma) = 0$, unless $\alpha + \beta + \gamma = 0$. However, it is $\varphi([\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k) \neq 0$ and thus it must be $\alpha + \beta \neq 0$ and $\alpha + \beta + \gamma = 0$. Thus $\xi_i + \xi_j + \xi_k = \kappa(\alpha) + \kappa(\beta) + \kappa(\gamma) = \kappa(\alpha + \beta + \gamma) = 0$. The converse is a trivial consequence of Lemma 1.3. $\square$

2. Symmetric $\mathfrak{t}$-triples for certain classes of flag manifolds

Important classes of generalized flag manifolds $M = G/K$ for which one can gives general expressions of symmetric $\mathfrak{t}$-triples are for instance, the flag manifolds G/K with second Betti number $b_2(G/K) = 1$, and the full flag manifolds G/T (which are such that $b_2(G/T) = \ell = \mathrm{rk}\, G$). We start with the first family.

2.1. Symmetric $\mathfrak{t}$-triples on flag manifolds with second Betti number equal to 1

Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G, defined by a subset $\Pi_M = \{\alpha_i\} \subset \Pi$. Then, it is $\dim_{\mathbb{R}} \mathfrak{t} = |\Pi_M| = 1$, thus M is such that $b_2(G/K) = 1$. In particular, any flag manifold with second Betti number equal to 1, is defined in this way. Recall that the *height* of a simple root $\alpha_i \in \Pi$ $(i = 1, \dots, \ell)$, is the positive integer m_i in the expression of the highest root $\widetilde{\alpha} = \sum_{k=1}^{\ell} m_k \alpha_k$ in terms of simple roots. We will denote by $\mathrm{ht} : \Pi \to \mathbb{Z}^+$ the function which associates to each simple root its height, that is $\mathrm{ht}(\alpha_i) = m_i$.

Theorem 2.1. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G, defined by a subset $\Pi_M = \{\alpha_i\}$ where the fixed simple root $\alpha_i \in \Pi$ is such that $\mathrm{ht}(\alpha_i) = r \geqslant 2$. Let $R_{\mathfrak{t}}$ be the associated $\mathfrak{t}$-root system and $\Pi_{\mathfrak{t}}$ the corresponding $\mathfrak{t}$-basis. Then, given $\xi \in R_{\mathfrak{t}}^+$ such that $\xi \notin \Pi_{\mathfrak{t}}$, it is $\xi - \bar{\alpha}_i \in R_{\mathfrak{t}}^+$, where $\bar{\alpha}_i = \kappa(\alpha_i) = \alpha_i|_{\mathfrak{t}} \in \Pi_{\mathfrak{t}}$.*

Proof. According to [7], a $\mathfrak{t}$-basis is given by $\Pi_{\mathfrak{t}} = \{\bar{\alpha}_i\}$ and thus $\mathfrak{t}^* = \mathbb{R}\bar{\alpha}_i$, where $\bar{\alpha}_i = \kappa(\alpha_i) = \alpha_i|_{\mathfrak{t}}$. However, by assumption it is $\mathrm{ht}(\alpha_i) = r \geqslant 2$, thus $|R_{\mathfrak{t}}^+| = r \geqslant 2$. Indeed, let $\alpha = \sum_{j=1}^{\ell} k_j \alpha_j \in R^+$, where the non-negative coefficients k_j are such that $k_j \leqslant m_j = \mathrm{ht}(\alpha_j)$ for any $j = 1, \dots, \ell$. Then we have $\kappa(\alpha) = k_i \bar{\alpha}_i$, with $1 \leqslant k_i \leqslant m_i = r$, which means that $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_i, 2\bar{\alpha}_i, \dots, r\bar{\alpha}_i\}$. Thus we obtain an irreducible decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \dots \oplus \mathfrak{m}_r$, where each summand $\mathfrak{m}_k$ is given by (3) for any $1 \leqslant k \leqslant r$. Let now $\xi \in R_{\mathfrak{t}}^+$ such that $\xi \neq \bar{\alpha}_i$. Then, it is $\xi = p\bar{\alpha}_i$ with $2 \leqslant p \leqslant r$ and so $\xi - \bar{\alpha}_i = p\bar{\alpha}_i - \bar{\alpha}_i = (p-1)\bar{\alpha}_i \in R_{\mathfrak{t}}^+$, which proves our claim. For $p = 2$ we have $\xi - \bar{\alpha}_i = \bar{\alpha}_i \in R_{\mathfrak{t}}^+$, while for $p = r$, it is $\xi - \bar{\alpha}_i = (r-1)\bar{\alpha}_i \in R_{\mathfrak{t}}^+$, since $r\bar{\alpha}_i \in R_{\mathfrak{t}}^+$. $\square$

Theorem 2.1 generalizes the following well-known theorem of root systems theory, to the $\mathfrak{t}$-root system $R_{\mathfrak{t}}$ corresponding to a flag manifold $M = G/K$ with $b_2(G/K) = 1$.

Lemma 2.2. *(See [19, p. 460].) Let R be the root system of a complex semi-simple Lie algebra. Choose a basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ and let R^+ be the induced ordering in R. If $\alpha \in R^+$ such that $\alpha \notin \Pi$, then there exists some $\alpha_i \in \Pi$ such that $\alpha - \alpha_i \in R^+$.*

Corollary 2.3. *Let $M = G/K$ be a flag manifold with $b_2(M) = 1$ and let $R_{\mathfrak{t}}$ be its $\mathfrak{t}$-root system. If $|R_{\mathfrak{t}}^+| \geqslant 2$, then any $\mathfrak{t}$-root $\xi \in R_{\mathfrak{t}}$ belongs to a symmetric $\mathfrak{t}$-triple, i.e. we can find $\zeta, \eta \in R_{\mathfrak{t}}$ such that $\xi + \zeta + \eta = 0$.*

Proof. Assume that M is defined by the subset $\Pi_M = \{\alpha_i\}$, for some fixed $i \in \{1, \dots, \ell\}$. Let $\xi \in R_{\mathfrak{t}}^+$. Then from Theorem 2.1 it is $\xi - \bar{\alpha}_i \in R_{\mathfrak{t}}^+$, where $\bar{\alpha}_i$ is the unique element of the associated $\mathfrak{t}$-basis $\Pi_{\mathfrak{t}}$. But then $(\bar{\alpha}_i, \xi - \bar{\alpha}_i, -\xi)$ is a symmetric $\mathfrak{t}$-triple since $\bar{\alpha}_i + (\xi - \bar{\alpha}_i) + (-\xi) = 0$ and $\bar{\alpha}_i, \xi - \bar{\alpha}_i \in R_{\mathfrak{t}}^+$, and $-\xi \in R_{\mathfrak{t}}^-$. Thus, a general form of symmetric $\mathfrak{t}$-triples on $M = G/K$ is

$$\pm(\bar{\alpha}_i, \xi - \bar{\alpha}_i, -\xi), \quad \xi \in R_{\mathfrak{t}}^+. \tag{4}$$

We call a symmetric $\mathfrak{t}$-triple of the form (4), a *symmetric $\mathfrak{t}$-triple of Type* A. Note that if $\xi \in \Pi_{\mathfrak{t}} \subset R_{\mathfrak{t}}^+$, i.e. $\xi = \bar{\alpha}_i$, then since by assumption it is $b_2(M) = 1$ and $|R_{\mathfrak{t}}^+| \geqslant 2$, the set $R_{\mathfrak{t}}^+$ must contain the $\mathfrak{t}$-roots $\xi = \bar{\alpha}_i$ and $2\xi = 2\bar{\alpha}_i$. Thus a symmetric $\mathfrak{t}$-triple which contains $\bar{\alpha}_i$ is given by $\pm(\bar{\alpha}_i, \bar{\alpha}_i, -2\bar{\alpha}_i)$. But this is also a symmetric $\mathfrak{t}$-triple of Type A since it is obtained from (4) for $\xi = 2\bar{\alpha}_i$. Now, if $\xi \in R_{\mathfrak{t}}^-$, then $-\xi \in R_{\mathfrak{t}}^+$, and thus in order to obtain the symmetric $\mathfrak{t}$-triple which contains $-\xi$ we have to replace in (4) ξ by $-\xi$. In this case the desired $\mathfrak{t}$-triple is given by $\pm(\bar{\alpha}_i, -\xi - \bar{\alpha}_i, \xi)$, which proves our claim. □

In the following, for a given flag manifold G/K with $\mathfrak{t}$-root system $R_{\mathfrak{t}}$, we will denote by $\mathcal{S}$ the set of inequivalent symmetric $\mathfrak{t}$-triples in $R_{\mathfrak{t}} \subset \mathfrak{t}^*$.

Theorem 2.4. *The only flag manifolds $M = G/K$ of a compact simple Lie group G, for which the set $\mathcal{S}$ is empty, are the compact isotropy irreducible Hermitian symmetric spaces.*

Proof. Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G and let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a system of simple roots for G. Assume that M is defined by a subset $\Pi_M = \{\alpha_i\} \subset \Pi$, for some fixed $i \in \{1, \dots, \ell\}$, such that $\mathrm{ht}(\alpha_i) = 1$. Then $M = G/K$ is an isotropy irreducible Hermitian symmetric space of compact type (cf. [19,14]). Indeed, a $\mathfrak{t}$-basis is given by $\Pi_{\mathfrak{t}} = \{\bar{\alpha}_i\}$ where $\bar{\alpha}_i = \kappa(\alpha_i) = \alpha_i|_{\mathfrak{t}}$. Since $\mathrm{ht}(\alpha_i) = 1$, it follows that $R_{\mathfrak{t}} = \{\pm\bar{\alpha}_i\}$ and $|R_{\mathfrak{t}}^+| = 1$, which means that M is isotropy irreducible. Due to the form of $R_{\mathfrak{t}}$ it is obvious that we cannot construct a symmetric $\mathfrak{t}$-triple which contains some of the $\mathfrak{t}$-roots $\bar{\alpha}_i$ or $-\bar{\alpha}_i$. Thus $\mathcal{S} = \emptyset$. □

Due to Theorem 2.4, it is now clear the condition $|R_{\mathfrak{t}}^+| \geqslant 2$ has been assumed in Theorem 2.1 and Corollary 2.3. In particular, any flag manifold $M = G/K$ of a compact simple Lie group G which is not an isotropy irreducible Hermitian symmetric space, is such that $|R_{\mathfrak{t}}^+| \geqslant 2$, and thus there exists at least one symmetric $\mathfrak{t}$-triple, of the form $(\xi, \zeta, -(\xi + \zeta))$, where $\xi \neq \zeta \in R_{\mathfrak{t}}^+$ (and thus, at least one non-zero structure constant).

Remark 2.5. The isotropy representation of a flag manifold $M = G/K$ of a compact simple Lie group G with $b_2(M) = 1$, may have at most six isotropy summands. This natural constraint comes from the form of the highest root $\tilde{\alpha}$ corresponding to a complex semi-simple Lie algebra (cf. [19, p. 477]). For example, a flag manifold $M = G/K$ with $b_2(M) = 1$ and $G \in \{\mathrm{SU}(\ell+1), \mathrm{SO}(2\ell+1), \mathrm{Sp}(\ell), \mathrm{SO}(2\ell)\}$, could be either an isotropy irreducible Hermitian symmetric space of compact type, or a generalized flag manifold with two isotropy summands. This is because the heights of the simple roots for classical simple Lie groups are not greater than two. Moreover, the only simple Lie group whose root system contains simple roots α_i with $\mathrm{ht}(\alpha_i) = 5$, or $\mathrm{ht}(\alpha_i) = 6$ is $G = \mathrm{E}_8$, and only for this Lie group we can determine flag manifolds $M = G/K$ with $b_2(M) = 1$ and five or six isotropy summands. For more details we refer to [14] (see also Example 2.11).

Remark 2.6. Since the $\mathfrak{t}$-root system of flag manifold $M = G/K$ as in Corollary 2.3, is given by $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_i, 2\bar{\alpha}_i, \dots, r\bar{\alpha}_i\}$, $r \geqslant 2$, another symmetric $\mathfrak{t}$-triple is given by

$$\pm\big(p\bar{\alpha}_i, q\bar{\alpha}_i, -(p+q)\bar{\alpha}_i\big), \tag{5}$$

where $2 \leqslant p, q \leqslant r$ such that $4 \leqslant p + q \leqslant r$. Since $p, q \geqslant 2$, it is obvious that these symmetric $\mathfrak{t}$-triples are inequivalent to the symmetric $\mathfrak{t}$-triples of Type A, and thus we will call them *symmetric $\mathfrak{t}$-triples of Type* B. Note that symmetric $\mathfrak{t}$-triples of Types A and B are the only possible symmetric $\mathfrak{t}$-triples which one can construct for a flag manifold $M = G/K$ with $b_2(M) = 1$, that is $\mathcal{S} = \{(\xi, \eta, \zeta)$ of Type A or B$\}$. By Remark 2.5 it follows that symmetric $\mathfrak{t}$-triples of Type B only exist for the values $r = 4, 5, 6$, and they are given in Table 1.

By Corollary 2.3, Remarks 2.5, 2.6 and Table 1 we also conclude that

Theorem 2.7. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group with $b_2(G/K) = 1$ and $|R_{\mathfrak{t}}^+| = r \geqslant 2$. Let N denote the number of non-equivalent symmetric $\mathfrak{t}$-triples on $\mathfrak{t}^*$, that is $N = |\mathcal{S}|$. Then $N \geqslant |R_{\mathfrak{t}}^+| - |\Pi_{\mathfrak{t}}| = r - 1$. The exact number N is given in Table 2.*

Table 1
Symmetric $\mathfrak{t}$-triples of Type B.

$r=4$	$r=5$	$r=6$
$\pm(2\bar{\alpha}_i, 2\bar{\alpha}_i, -4\bar{\alpha}_i)$	$\pm(2\bar{\alpha}_i, 2\bar{\alpha}_i, -4\bar{\alpha}_i)$ $\pm(2\bar{\alpha}_i, 3\bar{\alpha}_i, -5\bar{\alpha}_i)$	$\pm(2\bar{\alpha}_i, 2\bar{\alpha}_i, -4\bar{\alpha}_i)$ $\pm(2\bar{\alpha}_i, 3\bar{\alpha}_i, -5\bar{\alpha}_i)$ $\pm(2\bar{\alpha}_i, 4\bar{\alpha}_i, -6\bar{\alpha}_i)$ $\pm(3\bar{\alpha}_i, 3\bar{\alpha}_i, -6\bar{\alpha}_i)$

Table 2
The number $N=|\mathcal{S}|$ of inequivalent symmetric $\mathfrak{t}$-triples on $\mathfrak{t}^*$ for $M=G/K$ with $b_2(G/K)=1$.

$r=\lvert R_{\mathfrak{t}}^+\rvert$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
N	1	2	4	6	9

Proof. From Corollary 2.3 we obtain at least $|R_{\mathfrak{t}}^+|-|\Pi_{\mathfrak{t}}|=r-1$ symmetric $\mathfrak{t}$-triples of Type A given by (4). Thus $N\geqslant r-1$. In particular, for $r=2$ or 3, one can determine only symmetric $\mathfrak{t}$-triples of Type A on $\mathfrak{t}^*$, and thus the exact number N of these triples is $N=1$ and $N=2$, respectively. If $4\leqslant r\leqslant 6$, by Remark 2.6 we know that on $\mathfrak{t}^*$, there exist also the symmetric $\mathfrak{t}$-triples of Type B. In any case, the exact number n of symmetric $\mathfrak{t}$-triples of Type B is obtained from Table 1. Thus for $4\leqslant r\leqslant 6$ the number N is given by $N=r-1+n$, where $n=1,2,4$ for $r=4,5,6$, respectively. $\square$

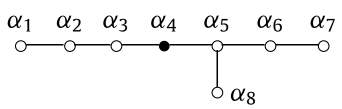
Next we present for all flag manifolds $M=G/K$ (of a compact simple Lie group G) with $b_2(G/K)=1$, the associated symmetric $\mathfrak{t}$-triples and the non-zero structure constants c_{ij}^k. We denote by $\Pi=\{\alpha_1,\dots,\alpha_\ell\}$ a basis of simple roots for G, adapted to the choice of G.

Example 2.8. Let $M=G/K$ defined by a subset $\Pi_M=\{\alpha_i\}\subset\Pi$ such that $\mathrm{ht}(\alpha_i)=2$. Then $R_{\mathfrak{t}}=\{\pm\bar{\alpha}_i,\pm2\bar{\alpha}_i\}$, $|R_{\mathfrak{t}}^+|=2$ and thus we obtain the decomposition $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2$, where the components $\mathfrak{m}_k$ are determined by (3). These spaces have been classified in [6] or [24]. By Theorem 2.7 we know that there exists only one symmetric $\mathfrak{t}$-triple of Type A, given by $\pm(\bar{\alpha}_i,\bar{\alpha}_i,-2\bar{\alpha}_i)$. Thus by Corollary 1.9 the only non-zero structure constant of $M=G/K$ is the triple c_{11}^2 and its symmetries. The number c_{11}^2 was calculated in [6] in terms of the dimensions $d_i=\dim_{\mathbb{R}}\mathfrak{m}_i$ $(i=1,2)$.

Example 2.9. Let $M=G/K$ defined by a subset $\Pi_M=\{\alpha_i\}\subset\Pi$ such that $\mathrm{ht}(\alpha_i)=3$. Then $R_{\mathfrak{t}}=\{\pm\bar{\alpha}_i,\pm2\bar{\alpha}_i,\pm3\bar{\alpha}_i\}$ and $|R_{\mathfrak{t}}^+|=3$. Thus, $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3$, where the summands $\mathfrak{m}_k$ are determined by (3). These flag manifolds have been classified in [20], but we note that they do not exhaust all flag manifolds with three isotropy summands (see Remark 2.12). By applying (4), we find two symmetric $\mathfrak{t}$-triples of Type A given by $\pm(\bar{\alpha}_i,\bar{\alpha}_i,-2\bar{\alpha}_i)$ and $\pm(\bar{\alpha}_i,2\bar{\alpha}_i,-3\bar{\alpha}_i)$. These are the only symmetric $\mathfrak{t}$-triples on $\mathfrak{t}^*$. Thus the non-zero triples are c_{11}^2, c_{12}^3 and their symmetries. The values of these triples can be found in [4].

Example 2.10. Let $M=G/K$ defined by a subset $\Pi_M=\{\alpha_i\}\subset\Pi$ such that $\mathrm{ht}(\alpha_i)=4$. Then $R_{\mathfrak{t}}=\{\pm\bar{\alpha}_i,\pm2\bar{\alpha}_i,\pm3\bar{\alpha}_i,\pm4\bar{\alpha}_i\}$ and $|R_{\mathfrak{t}}^+|=4$. Thus $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4$, where the components $\mathfrak{m}_k$ are given by (3). These spaces have been classified in [7], but they do not exhaust all flag manifolds with four isotropy summands (see Remark 2.13). By applying (4) we find three symmetric $\mathfrak{t}$-triples of Type A, given as follows: $\pm(\bar{\alpha}_i,\bar{\alpha}_i,-2\bar{\alpha}_i)$, $\pm(\bar{\alpha}_i,2\bar{\alpha}_i,-3\bar{\alpha}_i)$, $\pm(\bar{\alpha}_i,3\bar{\alpha}_i,-4\bar{\alpha}_i)$. Also, as we have seen in Table 1, by applying (5) we obtain a symmetric $\mathfrak{t}$-triple of Type B, given by $\pm(2\bar{\alpha}_i,2\bar{\alpha}_i,-4\bar{\alpha}_i)$. Thus the non-zero structure constants of $M=G/K$ are $c_{11}^2, c_{12}^3, c_{13}^4, c_{22}^4$, and their symmetries (cf. [7]).

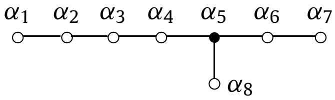
Example 2.11. Let $M=G/K$ defined by a subset $\Pi_M=\{\alpha_i\}\subset\Pi$ such that $\mathrm{ht}(\alpha_i)=5$. By Remark 2.5 we know that such a choice exists only for $G=\mathrm{E}_8$. In particular, a basis of simple roots for the root system of G_8 can be chosen such that the highest root $\tilde{\alpha}=2\alpha_1+3\alpha_2+4\alpha_3+5\alpha_4+6\alpha_5+4\alpha_6+2\alpha_7+3\alpha_8$ (cf. [2,7]). Thus, only the simple root α_4 is such that $\mathrm{ht}(\alpha_4)=5$ and by setting $\Pi_M=\{\alpha_4\}$ we obtain the painted Dynkin diagram



which defines the flag manifold $M=G/K=\mathrm{E}_8/\mathrm{U}(1)\times\mathrm{SU}(4)\times\mathrm{SU}(5)$. Here we have the decomposition $\mathfrak{m}=\mathfrak{m}_1\oplus\cdots\oplus\mathfrak{m}_5$, since $R_{\mathfrak{t}}^+=\{\bar{\alpha}_4,2\bar{\alpha}_4,3\bar{\alpha}_4,4\bar{\alpha}_4,5\bar{\alpha}_4\}$. Theorem 2.7 states that $|\mathcal{S}|=6$. Indeed, we obtain the following symmetric $\mathfrak{t}$-triples:

$$\bar{\alpha}_4+\bar{\alpha}_4+(-2\bar{\alpha}_4)=0,\qquad \bar{\alpha}_4+2\bar{\alpha}_4+(-3\bar{\alpha}_4)=0,\qquad \bar{\alpha}_4+3\bar{\alpha}_4+(-4\bar{\alpha}_4)=0,$$
$$\bar{\alpha}_4+4\bar{\alpha}_4+(-5\bar{\alpha}_4)=0,\qquad 2\bar{\alpha}_4+2\bar{\alpha}_4+(-4\bar{\alpha}_4)=0,\qquad 2\bar{\alpha}_4+3\bar{\alpha}_4+(-5\bar{\alpha}_4)=0.$$

Thus the non-zero structure constants are c_{11}^2, c_{12}^3, c_{13}^4, c_{14}^5, c_{22}^4, c_{23}^5, and their symmetries. Notice that if we set $\Pi_M = \{\alpha_5\}$, then we obtain the painted Dynkin diagram



which defines the flag manifold $M = G/K = \mathrm{E}_8/\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)\times\mathrm{SU}(5)$. Since $\mathrm{ht}(\alpha_5) = 6$, this is the only flag manifold (of a compact simple Lie group) with $b_2(M) = 1$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_6$. For this case, Theorem 2.7 states that $|\mathcal{S}| = 9$. Indeed, once can easily determine the following symmetric $\mathfrak{t}$-triples:

$$\bar\alpha_5 + \bar\alpha_5 + (-2\bar\alpha_5) = 0, \qquad \bar\alpha_5 + 2\bar\alpha_5 + (-3\bar\alpha_5) = 0, \qquad \bar\alpha_5 + 3\bar\alpha_5 + (-4\bar\alpha_5) = 0,$$
$$\bar\alpha_5 + 4\bar\alpha_5 + (-5\bar\alpha_5) = 0, \qquad \bar\alpha_5 + 5\bar\alpha_5 + (-6\bar\alpha_5) = 0, \qquad 2\bar\alpha_5 + 2\bar\alpha_5 + (-4\bar\alpha_5) = 0,$$
$$2\bar\alpha_5 + 3\bar\alpha_5 + (-5\bar\alpha_5) = 0, \qquad 2\bar\alpha_5 + 4\bar\alpha_5 + (-6\bar\alpha_5) = 0, \qquad 3\bar\alpha_5 + 3\bar\alpha_5 + (-6\bar\alpha_5) = 0.$$

Thus, the non-zero structure constants are c_{11}^2, c_{12}^3, c_{13}^4, c_{14}^5, c_{15}^6, c_{22}^4, c_{23}^5, c_{24}^6, c_{33}^6 and their symmetries.

Remark 2.12. In [20], it was proved that flag manifolds with three isotropy summands are also defined by setting $\Pi_M = \{\alpha_i, \alpha_j\colon\ i \neq j\}$ such that $\mathrm{ht}(\alpha_i) = \mathrm{ht}(\alpha_j) = 1$. For these spaces it was shown that there is only one non-zero structure constant, namely c_{12}^3 and its symmetries. Indeed, for such a flag manifold it is $\dim_{\mathbb{R}}\mathfrak{t} = 2$ and thus $b_2(M) = 2$. A $\mathfrak{t}$-basis is given by $\Pi_{\mathfrak{t}} = \{\bar\alpha_i = \alpha_i|_{\mathfrak{t}},\ \bar\alpha_j = \alpha_j|_{\mathfrak{t}}\colon\ i \neq j\}$, thus by choosing a positive root $\alpha = \sum_{k=1}^{\ell} c_k\alpha_k \in R_M^+$ we conclude that any positive $\mathfrak{t}$-root is given by $\xi = \kappa(\alpha) = c_i\bar\alpha_i + c_j\bar\alpha_j$, where $0 \leqslant c_1, c_j \leqslant 1$, since by assumption it is $\mathrm{ht}(\alpha_i) = \mathrm{ht}(\alpha_j) = 1$. Note that we cannot have simultaneously $c_i = c_j = 0$, because then $\alpha \in R_K$. Therefore the $\mathfrak{t}$-root system is given by $R_{\mathfrak{t}} = \{\pm\bar\alpha_i, \pm\bar\alpha_j, \pm(\bar\alpha_i + \bar\alpha_j)\}$, and so $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where the summands $\mathfrak{m}_k$ are defined according to (3). Now it is obvious that the only symmetric $\mathfrak{t}$-triple is given by $\pm(\bar\alpha_i, \bar\alpha_j, -(\bar\alpha_i + \bar\alpha_j))$ and thus $c_{12}^3 \neq 0$.

Remark 2.13. In [7], the author proved that flag manifolds with four isotropy summands are also defined by sets of the form $\Pi_M = \{\alpha_i, \alpha_j\colon\ i \neq j\}$ such that $\mathrm{ht}(\alpha_i) = 1$, $\mathrm{ht}(\alpha_j) = 2$, or $\mathrm{ht}(\alpha_i) = 2$, $\mathrm{ht}(\alpha_j) = 1$. However, since subsets Π_M of the last form also determine flag manifolds $M = G/K$ with five isotropy summands, this correspondence is not one-to-one. Indeed, a $\mathfrak{t}$-basis is given by $\Pi_{\mathfrak{t}} = \{\bar\alpha_i, \bar\alpha_j\colon\ i \neq j\}$, where $\bar\alpha_i = \alpha_i|_{\mathfrak{t}}$ and $\bar\alpha_j = \alpha_j|_{\mathfrak{t}}$. If we assume for example that $\mathrm{ht}(\alpha_i) = 1$, and $\mathrm{ht}(\alpha_j) = 2$, then given a root $\alpha = \sum_{k=1}^{\ell} c_k\alpha_k \in R^+$ we have $\xi = \kappa(\alpha) = c_i\bar\alpha_i + c_j\bar\alpha_j$, where $0 \leqslant c_1 \leqslant 1$ and $0 \leqslant c_j \leqslant 2$. Thus, one can determine at most five positive $\mathfrak{t}$-roots, given by $\bar\alpha_i$, $\bar\alpha_j$, $\bar\alpha_i + \bar\alpha_j$, $\bar\alpha_i + 2\bar\alpha_j$, $2\bar\alpha_j$. The existence of the last $\mathfrak{t}$-root in the above sequence depends on the root system of G, and more particularly on the existence of a root $\alpha = \sum_{k=1}^{\ell} c_k\alpha_k \in R^+$ such that $c_i = 0$ and $c_j = 2$. Such roots appear if $G \in \{\mathrm{SO}(2\ell+1), \mathrm{SO}(2\ell), \mathrm{E}_6, \mathrm{E}_7\}$, and the associated flag manifolds $M = G/K$ are such that $|R_{\mathfrak{t}}^+| = 5$ (cf. [7, Proposition 6]). The subsets $\Pi_M = \{\alpha_i, \alpha_j\colon\ i \neq j\}$ which define exactly four positive $\mathfrak{t}$-roots can be found in [7, Table 4]. In order to study the symmetric $\mathfrak{t}$-triples for the associated flag manifold $M = G/K$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$, we set for simplicity $\Pi_M^A = \{\alpha_i, \alpha_j\colon\ \mathrm{ht}(\alpha_i) = 1,\ \mathrm{ht}(\alpha_j) = 2\}$ and $\Pi_M^B = \{\alpha_i, \alpha_j\colon\ \mathrm{ht}(\alpha_i) = 2,\ \mathrm{ht}(\alpha_j) = 1\}$. Then, the corresponding $\mathfrak{t}$-root systems are given by $R_{\mathfrak{t}}^A = \{\pm\bar\alpha_i, \pm\bar\alpha_j, \pm(\bar\alpha_i + \bar\alpha_j), \pm(\bar\alpha_i + 2\bar\alpha_j)\}$ and $R_{\mathfrak{t}}^B = \{\pm\bar\alpha_i, \pm\bar\alpha_j, \pm(\bar\alpha_i + \bar\alpha_j), \pm(2\bar\alpha_i + \bar\alpha_j)\}$, respectively. For $R_{\mathfrak{t}}^A$ we find the symmetric $\mathfrak{t}$-triples $\pm(\bar\alpha_i, \bar\alpha_j, -(\bar\alpha_i + \bar\alpha_j))$, and $\pm(\bar\alpha_j, \bar\alpha_i + \bar\alpha_j, -(\bar\alpha_i + 2\bar\alpha_j))$. So, the only non-zero structure constants of the corresponding flag manifolds, are c_{12}^3, c_{23}^4 and their symmetries. For $R_{\mathfrak{t}}^B$ we get two symmetric $\mathfrak{t}$-triples given by $\pm(\bar\alpha_i, \bar\alpha_j, -(\bar\alpha_i + \bar\alpha_j))$, and $\pm(\bar\alpha_i, \bar\alpha_i + \bar\alpha_j, -(2\bar\alpha_i + \bar\alpha_j))$. Hence, the only non-zero structure constants of the corresponding flag manifolds, are c_{12}^3, c_{13}^4 and their symmetries. The explicit values of these triples are given in [7].

2.2. Symmetric $\mathfrak{t}$-triples for full flag manifolds $M = G/T$

Let us now extend our study of symmetric $\mathfrak{t}$-triples on full flag manifolds $M = G/T$, where T is maximal torus of a compact connected simple Lie group G. Such a space is obtained by painting black all nodes in the Dynkin diagram of G, that is $\Pi_K = \emptyset$ and $\Pi_M = \Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. It follows that $R_K = \emptyset$ and $R = R_M$, i.e. the set of complementary roots of M is identified with the root system of G, and hence the associated $\mathfrak{t}$-root system $R_{\mathfrak{t}}$ of G/T has the properties of a root system. It is obvious that $\mathfrak{t} = T_e(T)$, thus $\dim_{\mathbb{R}}\mathfrak{t} = \dim_{\mathbb{R}} T = \ell = \mathrm{rk}\, G$ and $b_2(G/T) = \ell$. Also $\mathfrak{t}^* = \sum_{i=1}^{\ell}\mathbb{R}\Lambda_i$, where Λ_i are the fundamental weights corresponding to Π, and a $\mathfrak{t}$-basis $\Pi_{\mathfrak{t}}$ is given by $\Pi_{\mathfrak{t}} = \{\bar\alpha_p = \alpha_p|_{\mathfrak{t}}\colon\ 1 \leqslant p \leqslant \ell\} = \{\bar\alpha_1, \ldots, \bar\alpha_\ell\}$, i.e. $|\Pi_{\mathfrak{t}}| = |\Pi_M| = |\Pi| = \ell$. Since $R_M^+ = R^+$, it is also $R_{\mathfrak{t}}^+ = \kappa(R^+)$ and $|R_{\mathfrak{t}}^+| = |R^+|$.

Proposition 2.14. *For a full flag manifold $M = G/T$ of a compact simple Lie group G, there is a bijective correspondence between roots and $\mathfrak{t}$-roots.*

Proof. The kernel of the linear map $\kappa : \mathfrak{h}^* \to \mathfrak{t}^*$ is given by $\mathrm{Ker}\,\kappa = \{\alpha \in \mathfrak{h}^*\colon\ \kappa(\alpha) = 0\} = R_K \cup \{0\}$, and thus in general κ is not an injection. However, for a full flag manifold $M = G/T$ it is $R_K = \emptyset$ and $\mathrm{Ker}\,\kappa = \{0\}$, thus we obtain the desired correspondence. $\square$

Proposition 2.15. *Let $M = G/T$ be a full flag manifold of a compact simple Lie group G. Then the isotropy representation of M decomposes into a direct sum of 2-dimensional pairwise inequivalent irreducible T-submodules $\mathfrak{m}_\alpha$. The number of these submodules is equal to $|R^+|$.*

Proof. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ be a reductive decomposition associated to G/T. Then $\mathfrak{m} = \sum_{\alpha \in R^+} (\mathfrak{m}_{\kappa(\alpha)} \oplus \mathfrak{m}_{-\kappa(\alpha)})^\tau = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha$, where we have set for simplicity $\mathfrak{m}_\alpha = (\mathfrak{m}_{\kappa(\alpha)} \oplus \mathfrak{m}_{-\kappa(\alpha)})^\tau$, for any $\alpha \in R^+$. Each $\mathfrak{m}_\alpha$ is a real irreducible $\mathrm{Ad}(T)$-submodule, which does not depend on the choice of the simple roots of R. From (3) it follows that $\mathfrak{m}_\alpha = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha = \mathbb{R}(E_\alpha + E_{-\alpha}) + \mathbb{R}i(E_\alpha - E_{-\alpha})$, and this completes the proof. □

In Table 1, following for example [19], we give for any full flag manifold G/T of a compact simple Lie group G the exact number of the corresponding isotropy summands.

The full flags $\mathrm{SU}(\ell+1)/T$ and $\mathrm{SO}(2\ell)/T$ for $\ell = 1$ and $\ell = 2$, respectively, give rise to the complex projective line $\mathbb{C}P^1 = \mathrm{SU}(2)/\mathrm{U}(1) \cong \mathrm{SO}(4)/\mathrm{U}(2)$, which is an isotropy irreducible Hermitian symmetric space and thus from Theorem 2.4 we have $\mathcal{S} = \emptyset$. As we will see in the following, this is the only full flag manifold for which one cannot define symmetric $\mathfrak{t}$-triples, and thus all structure constants are zero. Next, we will prove that full flag manifolds give rise to a second class of flag manifolds, for which a generalization of Lemma 2.2 holds.

Theorem 2.16. *Let $M = G/T$ be a full flag manifold of a compact simple Lie group G, $R_\mathfrak{t}$ the associated $\mathfrak{t}$-root system, and $\Pi_\mathfrak{t}$ the corresponding $\mathfrak{t}$-basis. Then, given $\xi \in R_\mathfrak{t}^+$ such that $\xi \notin \Pi_\mathfrak{t}$ we can find at least one $\alpha_p \in \Pi$ such that $\xi - \bar{\alpha}_p \in R_\mathfrak{t}^+$, where $\bar{\alpha}_p = \kappa(\alpha_p) = \alpha_p|_\mathfrak{t} \in \Pi_\mathfrak{t}$.*

Proof. We have seen that a $\mathfrak{t}$-basis is given by $\Pi_\mathfrak{t} = \{\bar{\alpha}_p = \alpha_p|_\mathfrak{t}\colon\ 1 \leqslant p \leqslant \ell\}$. Assume that the result is false, that is $\xi - \bar{\alpha}_p \notin R_\mathfrak{t}^+$, for any $\bar{\alpha}_p = \alpha_p|_\mathfrak{t} \in \Pi_\mathfrak{t}$. We will show that $\xi - \bar{\alpha}_p \notin R_\mathfrak{t}^-$, so $\xi - \bar{\alpha}_p \notin R_\mathfrak{t}$. In contrary, we assume that $\xi - \bar{\alpha}_p \in R_\mathfrak{t}^-$. Thus $\bar{\alpha}_p - \xi \in R_\mathfrak{t}^+$. But then $\bar{\alpha}_p = (\bar{\alpha}_p - \xi) + \xi$, which is a contradiction because $\bar{\alpha}_p$ is a simple $\mathfrak{t}$-root and it cannot be expressed as the sum of two positive $\mathfrak{t}$-roots. Therefore $\xi - \bar{\alpha}_p \notin R_\mathfrak{t}$. Let now α be a root such that $\kappa(\alpha) = \xi$. By assumption, it is $\xi \in R_\mathfrak{t}^+$ and $\xi \notin \Pi_\mathfrak{t}$. Since the projection κ always maps roots from Π_M to simple $\mathfrak{t}$-roots (see [7]), we conclude that it must be $\alpha \in R^+$ and $\alpha \notin \Pi$. However, we proved that $\xi - \bar{\alpha}_p \notin R_\mathfrak{t}$, or equivalently $\xi - \bar{\alpha}_p = \kappa(\alpha) - \kappa(\alpha_p) = \kappa(\alpha - \alpha_p) \notin R_\mathfrak{t}$. Since $R_K = \emptyset$ and $\kappa(0) = 0 \notin R_\mathfrak{t}$, where $0 \in \mathfrak{t}^*$, the last condition is true if and only if $\alpha = \alpha_p$, or $\alpha - \alpha_p \notin R$. The first condition is rejected by assumption. The second one is a contradiction due to Lemma 2.2. This proves our claim. □

Remark 2.17. Note that Theorem 2.16 does not tell us anything about the uniqueness of the simple root α_p, but just for its existence. Thus to a $\mathfrak{t}$-root $\xi \in R_\mathfrak{t}$ (with $\xi \notin \Pi_t$), there may correspond more that one simple roots $\alpha_p \in \Pi$ with $\xi - \bar{\alpha}_p \in R_\mathfrak{t}^+$ (see Example 2.21).

Corollary 2.18. *Let $M = G/T$ be a full flag manifold of a compact simple Lie group G with $\mathrm{rk}\,G = \dim_\mathbb{R} T \geqslant 2$, and let $R_\mathfrak{t}$ be the associated $\mathfrak{t}$-root system. Then any $\mathfrak{t}$-root $\xi \in R_\mathfrak{t}$ such that $\xi \notin \Pi_\mathfrak{t}$, belongs to a symmetric $\mathfrak{t}$-triple, i.e. we can always find $\zeta, \eta \in R_\mathfrak{t}$ such that $\xi + \zeta + \eta = 0$.*

Proof. Let $\xi \in R_\mathfrak{t}^+$. By Theorem 2.16 we can find a suitable simple root $\alpha_p \in \Pi = \Pi_M$, such that $\xi - \bar{\alpha}_p \in R_\mathfrak{t}^+$, where $\bar{\alpha}_p = \alpha_p|_\mathfrak{t} \in \Pi_\mathfrak{t}$. Then $(\bar{\alpha}_p, \xi - \bar{\alpha}_p, -\xi)$ is the desired symmetric $\mathfrak{t}$-triple, since $\bar{\alpha}_p + (\xi - \bar{\alpha}_p) + (-\xi) = 0$, $\bar{\alpha}_p, \xi - \bar{\alpha}_p \in R_\mathfrak{t}^+$, and $-\xi \in R_\mathfrak{t}^-$. Thus on G/T we can determine symmetric $\mathfrak{t}$-triples in the following way:

$$\pm(\bar{\alpha}_p, \xi - \bar{\alpha}_p, -\xi), \quad \xi \in R_\mathfrak{t}^+. \tag{6}$$

If ξ is negative then $-\xi \in R_\mathfrak{t}^+$, and thus in order to obtain the symmetric $\mathfrak{t}$-triple which contains $-\xi$ we have to replace in (6) ξ by $-\xi$. □

Note that other inequivalent symmetric $\mathfrak{t}$-triples in the $\mathfrak{t}$-root system $R_\mathfrak{t}$ of G/T, can be obtained as follows

$$\pm\big(\eta, \zeta, -(\eta + \zeta)\big), \tag{7}$$

where $\eta, \zeta \in R_\mathfrak{t}$ such that $\eta + \zeta \in R_\mathfrak{t}$, but $\xi, \zeta \notin \Pi_\mathfrak{t}$.

Corollary 2.19. *For a full flag manifold G/T of a compact simple Lie group G with $\mathrm{rk}\,G = \dim_\mathbb{R} T \geqslant 2$, there exist at least $|R^+| - |\Pi| = |R^+| - \ell$ inequivalent symmetric $\mathfrak{t}$-triples.*

Proof. This is an immediate consequence of Corollary 2.18. □

Remark 2.20. Note that the condition $\mathrm{rk}\,G = \dim_\mathbb{R} T \geqslant 2$ is assumed in order to avoid the extreme case of $\mathbb{C}P^1 = \mathrm{SU}(2)/\mathrm{U}(1) = \mathrm{SO}(4)/\mathrm{U}(2)$, for which we know that $\mathcal{S} = \emptyset$. About the first examples of full flag manifolds G/T, where G

Table 3
The number of the isotropy summands for G/T.

Simple Lie group G	Full flag manifold G/T	$\|R\| = \|R_{\mathfrak{t}}\|$	$\mathfrak{m} = \bigoplus_{i=1}^{s} \mathfrak{m}_i$
$\mathrm{SU}(\ell+1)$, $\ell \geqslant 1$	$\mathrm{SU}(\ell+1)/T$	$\ell(\ell+1)$	$s = \ell(\ell+1)/2$
$\mathrm{SO}(2\ell+1)$, $\ell \geqslant 2$	$\mathrm{SO}(2\ell+1)/T$	$2\ell^2$	$s = \ell^2$
$\mathrm{Sp}(\ell)$, $\ell \geqslant 2$	$\mathrm{Sp}(\ell)/T$	$2\ell^2$	$s = \ell^2$
$\mathrm{SO}(2\ell)$, $\ell \geqslant 3$	$\mathrm{SO}(2\ell)/T$	$2\ell(\ell-1)$	$s = \ell(\ell-1)$
G_2	G_2/T	12	$s = 6$
F_4	F_4/T	48	$s = 24$
E_6	E_6/T	72	$s = 36$
E_7	E_7/T	126	$s = 63$
E_8	E_8/T	240	$s = 120$

is one of the Lie groups $\mathrm{SU}(\ell+1)$, $\mathrm{SO}(2\ell+1)$, $\mathrm{Sp}(\ell)$ with $\ell \geqslant 2$, or $G = \mathrm{SO}(2\ell)$, with $\ell \geqslant 3$, we have $\mathcal{S} \neq \emptyset$. In particular, by computing the associated $\mathfrak{t}$-root systems, we see that

G	G/T	The non-zero structure constants
SU(3)	$\mathrm{SU}(3)/T$	c_{12}^{2} (see Remark 2.12)
SO(5)	$\mathrm{SO}(5)/T$	c_{12}^{3}, c_{23}^{4} (see Remark 2.13)
Sp(2)	$\mathrm{Sp}(2)/T$	c_{12}^{3}, c_{13}^{4} (see Remark 2.13)
SO(6)	$\mathrm{SO}(6)/T$	$c_{12}^{4}, c_{13}^{5}, c_{25}^{6}, c_{34}^{6}$

Example 2.21. Consider the full flag manifold $\mathrm{SU}(4)/T$. Recall that a maximal torus T of $\mathrm{SU}(4)$ is given by $T = \{\mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_4}\}\colon \theta_i \in \mathbb{R},\ \sum_i \theta_i = 0\}$, with Lie algebra $\mathfrak{t} = \{\mathrm{diag}\{i\theta_1, \ldots, i\theta_4\}\colon \theta_i \in \mathbb{R},\ \sum_i \theta_i = 0\}$. Thus the complexification $\mathfrak{t}^{\mathbb{C}} = \{\mathrm{diag}\{h_1, \ldots, h_4\}\colon h_i \in \mathbb{C},\ \sum_i h_i = 0\}$, is a Cartan subalgebra of the complex simple Lie algebra $(\mathfrak{su}(4))^{\mathbb{C}} = \mathfrak{sl}_4\mathbb{C}$. It is obvious that $\dim_{\mathbb{R}} \mathfrak{t} = \dim_{\mathbb{C}} \mathfrak{t}^{\mathbb{C}} = 3 = \mathrm{rk}(\mathrm{SU}(4))$. The root system of $\mathfrak{sl}_4\mathbb{C}$ with respect to $\mathfrak{t}^{\mathbb{C}}$, is given by $R = \{\pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_1 - e_4), \pm(e_2 - e_3), \pm(e_2 - e_4), \pm(e_3 - e_4)\}$, and a basis of simple roots can be chosen as follows: $\Pi = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4\}$. With respect to Π an ordering in R is given by $R^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. The full flag $M = \mathrm{SU}(4)/T$ is obtained by setting $\Pi_M = \Pi$, and we have $R_M = R$ and $R_M^+ = R^+$. Since $|\Pi_M| = 3$, it is obvious that $b_2(M) = 3$. A $\mathfrak{t}$-basis is given $\Pi_{\mathfrak{t}} = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3\}$ where $\bar{\alpha}_i = \alpha_i|_{\mathfrak{t}}$, for any $i \in \{1, 2, 3\}$, and the positive $\mathfrak{t}$-roots have the form $R_{\mathfrak{t}}^+ = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_1 + \bar{\alpha}_2, \bar{\alpha}_2 + \bar{\alpha}_3, \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3\}$. Thus $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_6$ (see also Table 3). Related to Remark 2.17, note that the positive $\mathfrak{t}$-root $\xi = \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3$ belongs into two inequivalent symmetric $\mathfrak{t}$-triples, since we can find two different simple roots $\alpha_p \neq \alpha_q$, such that $\xi - \bar{\alpha}_p \in R_{\mathfrak{t}}^+$ and $\xi - \bar{\alpha}_q \in R_{\mathfrak{t}}^+$. Indeed, we have $\xi - \bar{\alpha}_1 = \bar{\alpha}_2 + \bar{\alpha}_3 \in R_{\mathfrak{t}}^+$ and $\xi - \bar{\alpha}_3 = \bar{\alpha}_1 + \bar{\alpha}_2 \in R_{\mathfrak{t}}^+$. Thus we obtain the symmetric $\mathfrak{t}$-triples $\pm(\bar{\alpha}_1, \xi - \bar{\alpha}_1, -\xi) = \pm(\bar{\alpha}_1, \bar{\alpha}_2 + \bar{\alpha}_3, -(\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3))$, and $\pm(\bar{\alpha}_3, \xi - \bar{\alpha}_3, -\xi) = \pm(\bar{\alpha}_3, \bar{\alpha}_1 + \bar{\alpha}_2, -(\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3))$. Also we can define the symmetric $\mathfrak{t}$-triples $\pm(\bar{\alpha}_1, \bar{\alpha}_2, -(\bar{\alpha}_1 + \bar{\alpha}_2))$ and $\pm(\bar{\alpha}_2, \bar{\alpha}_3, -(\bar{\alpha}_2 + \bar{\alpha}_3))$. Thus the only non-zero structure constants are c_{12}^{3}, c_{23}^{5}, c_{15}^{6}, c_{34}^{6}, and their symmetries.

Remark 2.22. Due to Proposition 2.14 one can establish a bijective correspondence between symmetric $\mathfrak{t}$-triples in the ℓ-dimensional vector space $\mathfrak{t}^*$ and triples of roots with zero sum:

$$\left\{\begin{array}{c}\text{symmetric } \mathfrak{t}\text{-triples}\\ \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0\\ \bar{\alpha}, \bar{\beta}, \bar{\gamma} \in R_{\mathfrak{t}}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{triples of roots with zero sum}\\ \alpha + \beta + \gamma = 0\\ \alpha, \beta, \gamma \in R\end{array}\right\}. \tag{8}$$

This correspondence shows that Theorem 2.16 is the formulation of Lemma 2.2 in terms of $\mathfrak{t}$-roots. Also (8) make possible the rewriting of all results about symmetric $\mathfrak{t}$-triples on full flag manifolds, just in terms of roots. For example, based on Proposition 2.15, one can denote the triple associated to the submodules $\mathfrak{m}_\alpha$, $\mathfrak{m}_\beta$, and $\mathfrak{m}_\gamma$ of $\mathfrak{m}$, by $\left[\begin{smallmatrix} & \gamma & \\ \alpha & & \beta\end{smallmatrix}\right]$. Then

Corollary 2.23. *Let $M = G/T$ be a full flag manifold of a compact simple Lie group G and let $\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}_\alpha$ be the associated $(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{ad}(T)$-modules. Then, $\left[\begin{smallmatrix} & \gamma & \\ \alpha & & \beta\end{smallmatrix}\right] \neq 0$, if and only if $\alpha + \beta - \gamma = 0$.*

Proof. This result is based on the correspondence (8). Note that we have $\alpha + \beta - \gamma = 0$ instead of $\alpha + \beta + \gamma = 0$, since in the triple $\left[\begin{smallmatrix} & \gamma & \\ \alpha & & \beta\end{smallmatrix}\right]$ associated to the submodules $\mathfrak{m}_\alpha$, $\mathfrak{m}_\beta$ and $\mathfrak{m}_\gamma$, the roots α, β and γ are positive by assumption (see Proposition 2.15). $\square$

Example 2.24. Consider the full flag manifold G_2/T. The root system of the exceptional Lie group G_2 is given by $R = \{\mp(2e_2 + e_3), \mp(e_2 + 2e_3), \pm(e_2 - e_3), \mp(e_2 + e_3), \mp e_2, \mp e_3\}$ (cf. [2]). We fix a system of simple roots to be $\Pi = \{\alpha_1 = e_2 - e_3,\ \alpha_2 = -e_2\}$. With respect to Π the positive roots are given by

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}. \tag{9}$$

The maximal root is $\widetilde{\alpha} = 2\alpha_1 + 3\alpha_2$. The angle between α_1 and α_2 is $5\pi/6$ and we have $\|\alpha_1\| = \sqrt{3}\|\alpha_2\|$. We set $\Pi_M = \Pi$. It determines the full flag manifold G_2/T. The set of complementary roots R_M coincide with the root system R, i.e. $R = R_M$. A $\mathfrak{t}$-base is given by $\Pi_{\mathfrak{t}} = \{\bar{\alpha}_1, \bar{\alpha}_2\}$, thus for any $\alpha \in R^+$, we have $\xi = \kappa(\alpha) = c_1\bar{\alpha}_1 + c_2\bar{\alpha}_2 \in R^+_{\mathfrak{t}}$, where $0 \leqslant c_1 \leqslant 2$ and $0 \leqslant c_2 \leqslant 3$. Thus, the positive $\mathfrak{t}$-roots are given as follows: $R^+_{\mathfrak{t}} = \{\xi_1 = \bar{\alpha}_1,\ \xi_2 = \bar{\alpha}_2,\ \xi_3 = \bar{\alpha}_1 + \bar{\alpha}_2,\ \xi_4 = \bar{\alpha}_1 + 2\bar{\alpha}_2,\ \xi_5 = \bar{\alpha}_1 + 3\bar{\alpha}_2,\ \xi_6 = 2\bar{\alpha}_1 + 3\bar{\alpha}_2\}$. The isotropy representation $\mathfrak{m}$ of G_2/T decomposes into six inequivalent irreducible $\mathrm{ad}(\mathfrak{t})$-submodules, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$, where any isotropy summand $\mathfrak{m}_\alpha$ ($\alpha \in R^+$) admits the following expression:

$$\left.\begin{aligned} &\mathfrak{m}_1 = \mathbb{R}A_{\alpha_1} + \mathbb{R}B_{\alpha_1}, \quad \mathfrak{m}_3 = \mathbb{R}A_{\alpha_1+\alpha_2} + \mathbb{R}B_{\alpha_1+\alpha_2}, \quad \mathfrak{m}_5 = \mathbb{R}A_{\alpha_1+3\alpha_2} + \mathbb{R}B_{\alpha_1+3\alpha_2}, \\ &\mathfrak{m}_2 = \mathbb{R}A_{\alpha_2} + \mathbb{R}B_{\alpha_2}, \quad \mathfrak{m}_4 = \mathbb{R}A_{\alpha_1+2\alpha_2} + \mathbb{R}B_{\alpha_1+2\alpha_2}, \quad \mathfrak{m}_6 = \mathbb{R}A_{2\alpha_1+3\alpha_2} + \mathbb{R}B_{2\alpha_1+3\alpha_2}. \end{aligned}\right\} \tag{10}$$

Let now determine the non-zero structure constants $\left[{k \atop ij}\right]$ of G_2/T. Due to Corollary 2.23, it is sufficient to determine all triples of roots (α, β, γ) with zero sum, that is $\alpha + \beta + (-\gamma) = 0$, for example. By using relations (9) and (10), we get

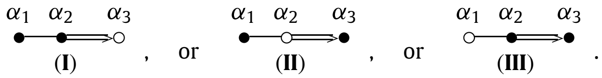
$$\begin{aligned} &\alpha_1 + \alpha_2 + \big(-(\alpha_1 + \alpha_2)\big) = 0, \qquad \alpha_1 + (\alpha_1 + 3\alpha_2) + \big(-(2\alpha_1 + 3\alpha_2)\big) = 0, \\ &\alpha_2 + (\alpha_1 + \alpha_2) + \big(-(\alpha_1 + 2\alpha_2)\big) = 0, \qquad (\alpha_1 + \alpha_2) + (\alpha_1 + 2\alpha_2) + \big(-(2\alpha_1 + 3\alpha_2)\big) = 0. \\ &\alpha_2 + (\alpha_1 + 2\alpha_2) + \big(-(\alpha_1 + 3\alpha_2)\big) = 0. \end{aligned}$$

Thus, the only non-zero structure constants of G_2/T are the following (for their values, see [11]):

$$\begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}, \quad \begin{bmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_2\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 23 \end{bmatrix}, \quad \begin{bmatrix} \alpha_1 + 3\alpha_2 \\ \alpha_2\ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 24 \end{bmatrix},$$
$$\begin{bmatrix} 2\alpha_1 + 3\alpha_2 \\ \alpha_1\ \alpha_1 + 3\alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}, \quad \begin{bmatrix} 2\alpha_1 + 3\alpha_2 \\ \alpha_1 + \alpha_2\ \alpha_1 + 2\alpha_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 34 \end{bmatrix}.$$

3. Homogeneous Einstein metrics on flag manifolds with five isotropy summands

Let us now proceed with the final part of the present work, the investigation of homogeneous Einstein metrics on flag manifolds G/K of the simple Lie group $G = \mathrm{SO}(7)$, whose isotropy representation decomposes into five pairwise inequivalent irreducible K-modules, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$. These spaces $M = \mathrm{SO}(7)/K$ are obtained by painting black two simple roots in the Dynkin diagram of SO(7). This means that $b_2(M) = 2$, but the converse is not true, that is, there is also a flag manifold $M = \mathrm{SO}(7)/K$ with $b_2(M) = 2$, with four isotropy summands. Indeed, by painting black two simple roots in the Dynkin diagram of SO(7) we get three possible cases:



The painted Dynkin diagrams (I), (II) and (III) determine the same coset $\mathrm{SO}(7)/K = \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$, with the difference that SU(2) is generated by the simple roots α_3, α_2 and α_1, respectively. The flag manifold which is defined by the PDD (I), is given also by $\mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SO}(3)$ and it belongs to the family $\mathrm{SO}(2\ell+1)/\mathrm{U}(1)^2 \times \mathrm{SO}(2\ell-3)$, which has four isotropy summands [7, Proposition 5]. Homogeneous Einstein metrics on this manifold have been classified in [7, Theorem 6]. The flag space which is defined by the PDD (II) is also presented by $\mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2)$ and it belongs to the family $\mathrm{SO}(2\ell+1)/\mathrm{U}(1) \times \mathrm{U}(\ell-1)$. This is defined by the subset $\Pi_M = \{\alpha_1, \alpha_\ell\} \subset \Pi$ of a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ corresponding to $\mathrm{SO}(2\ell+1)$, and it has five isotropy summands [7, Proposition 6]. The flag manifold defined by the PDD (III) has also five isotropy summands and it belongs to the family $\mathrm{SO}(2\ell+1)/\mathrm{U}(1)^2 \times \mathrm{SU}(\ell-1)$, which is defined by the set $\Pi_M = \{\alpha_{\ell-1}, \alpha_\ell\} \subset \Pi$.

3.1. The congruence of the flag manifolds defined by the PDD (II) and (III)

Next we will see that the flag manifolds $\mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2)$ and $\mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$, defined by the subsets $\Pi_M = \{\alpha_1, \alpha_3\}$ and $\Pi_M = \{\alpha_2, \alpha_3\}$, respectively, apart from diffeomorphic they are also isometric (as real manifolds). First we fix some notation. Let $R = \{\pm e_i \pm e_j\colon 1 \leqslant i \neq j \leqslant 3\} \cup \{\pm e_i\colon 1 \leqslant i \leqslant 3\}$ be the root system of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(7, \mathbb{C})$. We fix, once and for all, a system of simple roots $\Pi = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3\}$ for R, and we denote by $R^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}$ the corresponding set of positive roots. Note that $\varphi(\alpha_1, \alpha_1) = \varphi(\alpha_2, \alpha_2) = 2$, $\varphi(\alpha_3, \alpha_3) = 1$, $\varphi(\alpha_1, \alpha_2) = \varphi(\alpha_2, \alpha_3) = -1$, and $\varphi(\alpha_1, \alpha_3) = 0$ (cf. [13]). Recall also that the roots of $\mathfrak{so}(7, \mathbb{C})$ are divided into two classes with respect to their length, namely the long roots $\{\pm e_i \pm e_j\colon 1 \leqslant i \neq j \leqslant 3\}$, and the short roots $\{\pm e_i\colon 1 \leqslant i \leqslant 3\}$ (cf. [13, p. 147]). For convenience, we express them in terms of simple roots:

$$\left.\begin{aligned} &\text{positive long roots: } L = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}, \\ &\text{positive short roots: } S = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}. \end{aligned}\right\} \tag{11}$$

Table 4
The flag manifolds of SO(7) with five isotropy summands.

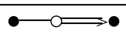
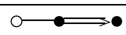
	PDD (II) 	PDD (III) 
(a)	$M=\mathrm{SO}(7)/\mathrm{U}(1)\times\mathrm{U}(2)$, $\mathfrak{m}=T_oM$	$M=\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SU}(2)$, $\mathfrak{n}=T_oM$
	$\Pi_M=\{\alpha_1,\alpha_3\}$, $\Pi_K=\{\alpha_2\}$, $R_K=\{\pm\alpha_2\}$ $R_M^+=\{\alpha_1,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_2+2\alpha_3,$ $\alpha_1+\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3,\alpha_1+2\alpha_2+2\alpha_3\}$ $\Pi_{\mathfrak{t}}=\{\bar\alpha_1=\alpha_1\vert_{\mathfrak{t}},\ \bar\alpha_3=\alpha_3\vert_{\mathfrak{t}}\}$ $R_{\mathfrak{t}}^+=\{\bar\alpha_1,\bar\alpha_3,\bar\alpha_1+\bar\alpha_3,2\bar\alpha_3,\bar\alpha_1+2\bar\alpha_3\}$	$\Pi_M=\{\alpha_2,\alpha_3\}$, $\Pi_K=\{\alpha_1\}$, $R_K=\{\pm\alpha_1\}$ $R_M^+=\{\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_2+2\alpha_3,$ $\alpha_1+\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3,\alpha_1+2\alpha_2+2\alpha_3\}$ $\Pi_{\mathfrak{t}}=\{\bar\alpha_2=\alpha_2\vert_{\mathfrak{t}},\ \bar\alpha_3=\alpha_3\vert_{\mathfrak{t}}\}$ $R_{\mathfrak{t}}^+=\{\bar\alpha_2,\bar\alpha_3,\bar\alpha_2+\bar\alpha_3,\bar\alpha_2+2\bar\alpha_3,2\bar\alpha_2+2\bar\alpha_3\}$
(b)	$\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5$	$\mathfrak{n}=\mathfrak{n}_1\oplus\mathfrak{n}_2\oplus\mathfrak{n}_3\oplus\mathfrak{n}_4\oplus\mathfrak{n}_5$
	$\mathfrak{m}_1=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_1}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{m}_2=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{m}_3=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_1+\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{m}_4=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=2\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{m}_5=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_1+2\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$	$\mathfrak{n}_1=\sum_{\alpha\in R_M^+\kappa(\alpha)=\bar\alpha_2}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{n}_2=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{n}_3=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_2+\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{n}_4=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=\bar\alpha_2+2\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$ $\mathfrak{n}_5=\sum_{\alpha\in R_M^+:\ \kappa(\alpha)=2\bar\alpha_2+2\bar\alpha_3}(\mathbb{R}A_\alpha+\mathbb{R}B_\alpha)$
(c)	Symmetric $\mathfrak{t}$-triples	Symmetric $\mathfrak{t}$-triples
	$(\bar\alpha_1,\bar\alpha_3,-(\bar\alpha_1+\bar\alpha_3))$ $(\bar\alpha_3,\bar\alpha_3,-2\bar\alpha_3)$ $(\bar\alpha_1,2\bar\alpha_3,-(\bar\alpha_1+2\bar\alpha_3))$ $(\bar\alpha_3,\bar\alpha_1+\bar\alpha_3,-(\bar\alpha_1+2\bar\alpha_3))$	$(\bar\alpha_2,\bar\alpha_3,-(\bar\alpha_2+\bar\alpha_3))$ $(\bar\alpha_3,\bar\alpha_2+\bar\alpha_3,-(\bar\alpha_2+2\bar\alpha_3))$ $(\bar\alpha_2,\bar\alpha_2+2\bar\alpha_3,-(2\bar\alpha_2+2\bar\alpha_3))$ $(\bar\alpha_2+\bar\alpha_3,\bar\alpha_2+\bar\alpha_3,-(2\bar\alpha_2+2\bar\alpha_3))$
(d)	Non-zero structure constants	Non-zero structure constants
	c_{12}^3, c_{22}^4, c_{14}^5, c_{23}^5	c_{12}^3, c_{23}^4, c_{14}^5, c_{33}^5

Table 5
The isotropy summands and their type with respect to the length of roots.

Isotropy decomposition $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5$	Type or roots	$d_i=\dim\mathfrak{m}_i$
$\mathfrak{m}_1=\mathrm{span}\{A_{\alpha_1}+B_{\alpha_1},A_{\alpha_1+\alpha_2}+B_{\alpha_1+\alpha_2}\}$	long	4
$\mathfrak{m}_2=\mathrm{span}\{A_{\alpha_3}+B_{\alpha_3},A_{\alpha_2+\alpha_3}+B_{\alpha_2+\alpha_3}\}$	short	4
$\mathfrak{m}_3=\mathrm{span}\{A_{\alpha_1+\alpha_2+\alpha_3}+B_{\alpha_1+\alpha_2+\alpha_3}\}$	short	2
$\mathfrak{m}_4=\mathrm{span}\{A_{\alpha_2+2\alpha_3}+B_{\alpha_2+2\alpha_3}\}$	long	2
$\mathfrak{m}_5=\mathrm{span}\{A_{\alpha_1+\alpha_2+2\alpha_3}+B_{\alpha_1+\alpha_2+2\alpha_3},A_{\alpha_1+2\alpha_2+2\alpha_3}+B_{\alpha_1+2\alpha_2+2\alpha_3}\}$	long	4
Isotropy decomposition $\mathfrak{n}=\mathfrak{n}_1\oplus\mathfrak{n}_2\oplus\mathfrak{n}_3\oplus\mathfrak{n}_4\oplus\mathfrak{n}_5$	Type of roots	$d_i'=\dim\mathfrak{n}_i$
$\mathfrak{n}_1=\mathrm{span}\{A_{\alpha_2}+B_{\alpha_2},A_{\alpha_1+\alpha_2}+B_{\alpha_1+\alpha_2}\}$	long	4
$\mathfrak{n}_2=\mathrm{span}\{A_{\alpha_3}+B_{\alpha_3}\}$	short	2
$\mathfrak{n}_3=\mathrm{span}\{A_{\alpha_2+\alpha_3}+B_{\alpha_2+\alpha_3},A_{\alpha_1+\alpha_2+\alpha_3}+B_{\alpha_1+\alpha_2+\alpha_3}\}$	short	4
$\mathfrak{n}_4=\mathrm{span}\{A_{\alpha_1+\alpha_2+2\alpha_3}+B_{\alpha_1+\alpha_2+2\alpha_3},A_{\alpha_2+2\alpha_3}+B_{\alpha_2+2\alpha_3}\}$	long	4
$\mathfrak{n}_5=\mathrm{span}\{A_{\alpha_1+2\alpha_2+2\alpha_3}+B_{\alpha_1+2\alpha_2+2\alpha_3}\}$	long	2

In Table 4 we present the most important features of the flag manifolds $\mathrm{SO}(7)/K$ defined by the PDD (II) and (III). In part (a), we describe the root system R_K of the semi-simple part $\mathfrak{su}(2,\mathbb{C})$ of $\mathfrak{k}^{\mathbb{C}}=\mathfrak{su}(2,\mathbb{C})\oplus\mathbb{C}\oplus\mathbb{C}$, the associated positive complementary roots R_M^+, a $\mathfrak{t}$-basis $\Pi_{\mathfrak{t}}$, and the corresponding positive $\mathfrak{t}$-root system $R_{\mathfrak{t}}^+$. In part (b) we state the reductive decomposition of $\mathfrak{g}=T_e\,\mathrm{SO}(7)$ with respect to the $\mathrm{Ad}(K)$-invariant inner product $(\cdot,\cdot)=-\varphi(\cdot,\cdot)$, and we apply relation (3) to determine the corresponding irreducible $\mathrm{ad}(\mathfrak{k})$-submodules. In part (c) we present the corresponding symmetric $\mathfrak{t}$-triples and finally, in part (d), we give the associated non-zero structure constants c_{ij}^k.

Proposition 3.1. *The flag manifolds* $\mathrm{SO}(7)/\mathrm{U}(1)\times\mathrm{U}(2)\cong\mathrm{SO}(7)/\mathrm{U}(1)^2\times\mathrm{SU}(2)$ *defined by the painted Dynkin diagrams* (II) *and* (III), *respectively, are isometric* (*as real manifolds*)*.*

Proof. By combining parts (a) and (b) of Table 4 and by using relation (11), we have established in Table 5 a correspondence between the associated isotropy summands and the type or roots generating them (with respect to their length). We also give the dimensions of these submodules. From the first and the third column of Table 5, it obvious that in order to prove our claim is sufficient to construct the following isometries:

$$\sigma:\mathfrak{m}_1\to\mathfrak{n}_1,\qquad\sigma:\mathfrak{m}_2\to\mathfrak{n}_3,\qquad\sigma:\mathfrak{m}_3\to\mathfrak{n}_2,\qquad\sigma:\mathfrak{m}_4\to\mathfrak{n}_5,\qquad\sigma:\mathfrak{m}_5\to\mathfrak{n}_4.$$

Then we will have establish the desired isometry $\sigma:\mathfrak{m}\to\mathfrak{n}$. This map $\sigma:\mathfrak{m}\to\mathfrak{n}$ is induced by the Weyl group $\mathcal{W}$ of $\mathfrak{so}(7,\mathbb{C})$, which acts on the root system R via the reflections $\{s_\alpha:\ \alpha\in R\}$, given by $s_\alpha(\beta)=\beta-\frac{2\varphi(\beta,\alpha)}{\varphi(\alpha,\alpha)}\alpha$, for any $\alpha,\beta\in R$. In particular, since the irreducible modules of the following pairs $(\mathfrak{m}_1,\mathfrak{n}_1)$, $(\mathfrak{m}_2,\mathfrak{n}_3)$, $(\mathfrak{m}_3,\mathfrak{n}_2)$, $(\mathfrak{m}_4,\mathfrak{n}_5)$, and $(\mathfrak{m}_5,\mathfrak{n}_4)$, are generated in any case by root vectors corresponding to roots of the same length, the existence of such an isometry between them, is a subsequence of the transitive action of $\mathcal{W}$ on roots of a given length. This means, that if we have a pair $(\mathfrak{m}_i,\mathfrak{n}_j)$

with $\dim \mathfrak{m}_i = \dim \mathfrak{n}_j$ and we assume for example that the modules $\mathfrak{m}_i$ and $\mathfrak{n}_j$ are generating by the vectors $A_\alpha + B_\alpha$ and $A_\beta + B_\beta$, respectively, where both α, β are such that $\alpha, \beta \in L$, then we can always find an element $w \in \mathcal{W}$ such that $w(\alpha) = \beta$ (or $w(\alpha) = -\beta$). Since an element of $\mathcal{W}$ induces inner automorphism of our Lie algebra, the element $w \in \mathcal{W}$ will determine an isometry $w : \mathfrak{m}_i \to \mathfrak{n}_j$ (the same is true for the short roots, too). Indeed, recall that the group $\mathcal{W}$ is generated by the simple reflections $\{s_1 = s_{\alpha_1} = s_{e_1 - e_2},\ s_2 = s_{\alpha_2} = s_{e_2 - e_3},\ s_3 = s_{\alpha_3} = s_{e_3}\}$. For these reflections it is $s_{\alpha_i}(\alpha_j) = s_i(\alpha_j) = \alpha_j - b_{ij}\alpha_i$, where $B = (b_{ij})$ is the transpose of the Cartan matrix of $\mathfrak{so}(7, \mathbb{C})$, given by $A = (A_{ij}) = (2\varphi(\alpha_i, \alpha_j)/\varphi(\alpha_j, \alpha_j)) = \left(\begin{smallmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{smallmatrix}\right)$. Thus, it is $s_1(\alpha_1) = -\alpha_1$, $s_2(\alpha_1) = \alpha_1 + \alpha_2$, $s_3(\alpha_1) = \alpha_1$, $s_1(\alpha_2) = \alpha_1 + \alpha_2$, $s_2(\alpha_2) = -\alpha_2$, $s_3(\alpha_2) = \alpha_2 + 2\alpha_3$, $s_1(\alpha_3) = \alpha_3$, $s_2(\alpha_3) = \alpha_2 + \alpha_3$, $s_3(\alpha_3) = -\alpha_3$ and we easily compute that

$$(s_2 \circ s_1)(\alpha_1) = -(\alpha_1 + \alpha_2), \tag{12}$$

$$(s_2 \circ s_1)(\alpha_1 + \alpha_2) = -\alpha_2, \tag{13}$$

$$(s_2 \circ s_1)(\alpha_3) = \alpha_2 + \alpha_3, \tag{14}$$

$$(s_2 \circ s_1)(\alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3, \tag{15}$$

$$(s_2 \circ s_1)(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_3, \tag{16}$$

$$(s_2 \circ s_1)(\alpha_2 + 2\alpha_3) = \alpha_1 + 2\alpha_2 + 2\alpha_3, \tag{17}$$

$$(s_2 \circ s_1)(\alpha_1 + \alpha_2 + 2\alpha_3) = \alpha_2 + 2\alpha_3, \tag{18}$$

$$(s_2 \circ s_1)(\alpha_1 + 2\alpha_2 + 2\alpha_3) = \alpha_1 + \alpha_2 + 2\alpha_3. \tag{19}$$

The relations (12)–(13) show that $\mathfrak{m}_1$, $\mathfrak{n}_1$ are isometric via the reflection $s_2 \circ s_1 : \mathfrak{m}_1 \to \mathfrak{n}_1$. Similar, relations (14)–(15) ensure that $s_2 \circ s_1$ maps $\mathfrak{m}_2 \to \mathfrak{n}_3$, relation (16) implies that $s_2 \circ s_1 : \mathfrak{m}_3 \to \mathfrak{n}_2$, and relation (17) implies that $s_2 \circ s_1 : \mathfrak{m}_4 \to \mathfrak{n}_5$. Finally, by (18)–(19) we conclude that $\mathfrak{m}_5$, $\mathfrak{n}_4$ are also isometric via the composition $s_2 \circ s_1$ and in this way we have define an isometry $\sigma = s_2 \circ s_1 : \mathfrak{m} \to \mathfrak{n}$. Note that under the above isometry $s_2 \circ s_1 : \mathfrak{m} \to \mathfrak{n}$, the symmetric $\mathfrak{t}$-triples and hence the structure constants of the flag manifolds defined by the PDD (II) and (III), are identified. $\square$

3.2. *Homogeneous Einstein metrics on* $\mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2) \cong \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$

For the construction of the Einstein equation for an SO(7)-invariant Riemannian metric we need to recall the explicit formulae of the Ricci tensor and the scalar curvature for a flag manifold $M = G/K$ in the general case.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to $(\cdot,\cdot) = -\varphi(\cdot,\cdot)$. As we have said before, a G-invariant Riemannian metric g on $M = G/K$ is identified with an $\mathrm{Ad}(K)$-invariant (or $\mathrm{ad}(\mathfrak{k})$-invariant) inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$. This inner product can be written by $\langle X, Y\rangle = (AX, Y)$ $(X, Y \in \mathfrak{m})$ for some $\mathrm{Ad}(K)$-invariant positive definite symmetric endomorphism $A : \mathfrak{m} \to \mathfrak{m}$. Due to Proposition 1.2, we can express A by the equation $A = \sum_{\xi \in R_{\mathfrak{t}}^+} x_\xi \cdot \mathrm{Id}|_{(\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau}$, where each element $\{x_\xi :\ \xi \in R_{\mathfrak{t}}^+\}$ is an eigenvalue (a positive real number) of A, with associated eigenspace the real $\mathrm{ad}(\mathfrak{k})$-module $(\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau$. If we assume for simplicity that $\mathfrak{m} = \bigoplus_{i=1}^s \mathfrak{m}_i$ is an $(\cdot,\cdot)$-orthogonal decomposition of $\mathfrak{m}$ into s pairwise inequivalent irreducible $\mathrm{ad}(\mathfrak{k})$-modules $\mathfrak{m}_i$ which are given by (3), then A is given by $A = \sum_{\xi_i \in R_{\mathfrak{t}}^+} x_{\xi_i} \cdot \mathrm{Id}|_{\mathfrak{m}_i} = \sum_{i=1}^s x_i \cdot \mathrm{Id}|_{\mathfrak{m}_i}$, where $x_i \equiv x_{\xi_i}$ for any $\xi_i \in R_{\mathfrak{t}}^+ = \{\xi_1, \ldots, \xi_s\}$. Due to the decomposition $\mathfrak{m}_i = \sum_{\alpha \in R_M^+:\ \kappa(\alpha) = \xi_i} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha)$, it is obvious that the vectors $\{A_\alpha, B_\alpha :\ \alpha \in R_M^+\}$ are eigenvectors of A, corresponding to the eigenvalue $x_i \equiv x_{\xi_i}$, and thus we also denote this eigenvalue by $x_\alpha \in \mathbb{R}^+$, where $\alpha \in R_M^+$ is such that $\kappa(\alpha) = \xi_i$, for any $1 \leqslant i \leqslant s$. As usual, we extend A to a complex linear operator in $\mathfrak{m}^{\mathbb{C}}$ without any change in notation. Hence the inner product $g = \langle\cdot,\cdot\rangle$ admits a natural extension to an $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariant bilinear symmetric form on $\mathfrak{m}^{\mathbb{C}}$, and for this one we maintain the same notation too. Then, the root vectors $\{E_\alpha :\ \alpha \in R_M\}$ are eigenvectors of $A : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ corresponding to the eigenvalues $x_\alpha = x_{-\alpha} > 0$. Note that $x_\alpha = x_\beta$ whenever $\alpha|_{\mathfrak{t}} = \beta|_{\mathfrak{t}}$, for any $\alpha, \beta \in R_M^+$. By the $\mathrm{ad}(\mathfrak{k}^{\mathbb{C}})$-invariance of the metric and the part (ii) of the definition of an invariant ordering, we also obtain $x_\alpha = x_{\alpha+\beta}$ for any $\alpha, \alpha + \beta \in R_M^+$ with $\beta \in R_K$.

The set of G-invariant Riemannian metrics on G/K is parametrized by $|R_{\mathfrak{t}}^+|$ real numbers, it means that any G-invariant Riemannian metric $g = \langle\cdot,\cdot\rangle = (A\,\cdot,\cdot)$ on G/K is given by

$$g = \langle\cdot,\cdot\rangle = (A\,\cdot,\cdot) = x_1 \cdot (\cdot,\cdot)|_{\mathfrak{m}_1} + \cdots + x_s \cdot (\cdot,\cdot)|_{\mathfrak{m}_s},$$

where $x_1 \equiv x_{\xi_1} > 0, \ldots, x_s \equiv x_{\xi_s} > 0$. Next, we will denote such an invariant Riemannian metric by $g = (x_1, \ldots, x_s) \in \mathbb{R}_+^s$. Note that given a G-invariant complex structure J on $M = G/K$ (such a structure is induced by an invariant ordering R_M^+ and it is determined by an $\mathrm{Ad}(K)$-invariant endomorphism $J_o : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ such that $J_o E_{\pm a} = \pm i E_{\pm \alpha}$ for any $\alpha \in R_M^+$), a G-invariant metric is Kähler with respect to J, if and only if the positive real numbers x_ξ satisfy the equation $x_{\xi+\zeta} = x_\xi + x_\zeta$ for any $\xi, \zeta, \xi + \zeta \in R_{\mathfrak{t}}^+ = \kappa(R_M^+)$. In other words, g is Kähler, if and only if $x_{\alpha+\beta} = x_\alpha + x_\beta$, where $\alpha, \beta, \alpha + \beta \in R_M^+$ are such that $\kappa(\alpha) = \xi$ and $\kappa(\beta) = \zeta$ (cf. [1]).

In view of the decomposition $\mathfrak{m} = \bigoplus_{k=1}^s \mathfrak{m}_k$, the Ricci tensor Ric_g of $(G/K, g = (x_1, \ldots, x_s))$, as a G-invariant symmetric covariant 2-tensor on G/K, is identified with an $\mathrm{Ad}(K)$-invariant symmetric bilinear form on $\mathfrak{m}$ and it is given by $\mathrm{Ric}_g =$

$\sum_{k=1}^{s} y_k \cdot (\cdot,\cdot)|_{\mathfrak{m}_k}$, for some $y_1,\dots,y_s \in \mathbb{R}$. Let $\{e_j^{(k)}\}_{j=1}^{d_k}$ be an $(\cdot,\cdot)$-orthonormal basis on $\mathfrak{m}_k$, where $d_k = \dim \mathfrak{m}_k$, for any $k \in \{1,\dots,s\}$. Then the set $\{X_j^{(k)} = 1/\sqrt{x_k} e_j^{(k)}\}$ is an $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{m}_k$, where $\langle\cdot,\cdot\rangle$ is the $\mathrm{Ad}(K)$-invariant inner product induced by the G-invariant metric $g = (x_1,\dots,x_s)$ of M. Let r_k be the real numbers defined by the equation $r_k = \mathrm{Ric}_g(X_j^{(k)}, X_j^{(k)}) = \mathrm{Ric}_g(\frac{e_j^{(k)}}{\sqrt{x_k}}, \frac{e_j^{(k)}}{\sqrt{x_k}}) = (1/x_k)\,\mathrm{Ric}_g(e_j^{(k)}, e_j^{(k)})$, that is $\mathrm{Ric}_g(e_j^{(k)}, e_j^{(k)}) = x_k r_k$. Then it obvious that we can express the Ricci tensor by $\mathrm{Ric}_g = \sum_{k=1}^{s}(x_k r_k)\cdot(\cdot,\cdot)|_{\mathfrak{m}_k}$. In particular, we have that

Proposition 3.2. *(See [22,26].) The components r_k of the Ricci tensor of $(G/K, g = (x_1,\dots,x_s))$ are given by $r_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{i,j}\frac{x_k}{x_i x_j}\left[{k \atop ij}\right] - \frac{1}{2d_k}\sum_{i,j}\frac{x_j}{x_k x_i}\left[{j \atop ki}\right]$, for any $k = 1,\dots,s$. Moreover, the scalar curvature S_g of $(G/K, g = (x_1,\dots,x_s))$ has the form $S_g = \operatorname{tr}\mathrm{Ric}_g = \sum_{k=1}^{s} d_k r_k = \frac{1}{2}\sum_{k=1}^{s}\frac{d_k}{x_k} - \frac{1}{4}\sum_{i,j,k}\left[{k \atop ij}\right]\frac{x_k}{x_i x_j}$.*

A G-invariant metric on $M = G/K$ is Einstein with Einstein constant $\lambda \in \mathbb{R}_+$, if and only if, it is a positive real solution of the system $\{r_1 = \lambda,\ r_2 = \lambda,\ \dots,\ r_k = \lambda\} \Leftrightarrow \{r_1 - r_2 = 0,\ r_2 - r_3 = 0,\ \dots,\ r_{k-1} - r_k = 0\}$.

Let us now focus again on the flag manifold $M = \mathrm{SO}(7)/\mathrm{U}(1)\times\mathrm{U}(2) \cong \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$. Due to the isometry $\sigma : \mathfrak{m} \to \mathfrak{n}$ presented in Section 3.1, we will not distinguish these SO(7)-flag spaces. Without loss of generality, we assume that M is defined for example by the PDD (II), that is $\Pi_M = \{\alpha_1, \alpha_3\}$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$. We consider SO(7)-invariant Riemannian metrics $g = \langle\cdot,\cdot\rangle$ on M, given by $g = \langle\cdot,\cdot\rangle = x_1\cdot(\cdot,\cdot)|_{\mathfrak{m}_1} + x_2\cdot(\cdot,\cdot)|_{\mathfrak{m}_2} + x_3\cdot(\cdot,\cdot)|_{\mathfrak{m}_3} + x_4\cdot(\cdot,\cdot)|_{\mathfrak{m}_4} + x_5\cdot(\cdot,\cdot)|_{\mathfrak{m}_5}$, where $(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}_+^5$. The Ricci tensor Ric_g of (M,g), with respect to an $\langle\cdot,\cdot\rangle$-orthonormal basis of $\mathfrak{m}$ is given by $\mathrm{Ric}_g = \sum_{k=1}^{5}(x_k r_k)\cdot(\cdot,\cdot)|_{\mathfrak{m}_k}$, where the components r_k are defined by Proposition 3.2. By using Table 5 we easily get:

Proposition 3.3. *The components r_k of the Ricci tensor Ric_g on $(M, g = (x_1,x_2,x_3,x_4,x_5))$ are given by*

$$\left.\begin{aligned}
r_1 &= \frac{1}{2x_1} + \frac{c_{12}^3}{2d_1}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + \frac{c_{14}^5}{2d_1}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right),\\
r_2 &= \frac{1}{2x_2} + \frac{c_{12}^3}{2d_2}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_1x_2}\right) + \frac{c_{23}^5}{2d_2}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right) - \frac{c_{22}^4}{2d_2}\frac{x_4}{x_2^2},\\
r_3 &= \frac{1}{2x_3} + \frac{c_{12}^3}{2d_3}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3}\right) + \frac{c_{23}^5}{2d_3}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_2x_3}\right),\\
r_4 &= \frac{1}{2x_4} + \frac{c_{14}^5}{2d_4}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right) + \frac{c_{22}^4}{4d_4}\left(\frac{x_4}{x_2^2} - \frac{2}{x_4}\right),\\
r_5 &= \frac{1}{2x_5} + \frac{c_{14}^5}{2d_5}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) + \frac{c_{23}^5}{2d_5}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right).
\end{aligned}\right\} \tag{20}$$

For the computation of the non-zero triples c_{12}^3, c_{14}^5, c_{22}^4 and c_{23}^5 we will use a Kähler–Einstein metric of M. For a detailed description of invariant complex structures and Kähler–Einstein metrics see [3] or [7]. Here we only recall that following well-known result:

Theorem 3.4. *(See [1,3,7].) Let J be the G-invariant complex structure on M defined by the invariant ordering R_M^+ in R_M. Then, the $\mathrm{ad}(\mathfrak{k})^{\mathbb{C}}$-invariant Riemannian metric on $\mathfrak{m}^{\mathbb{C}}$ given by $g_J = \{x_\alpha = c\cdot\varphi(\delta_{\mathfrak{m}},\alpha)\ (c\in\mathbb{R})\colon \alpha \in R_M^+\}$, where $\delta_{\mathfrak{m}} = \frac{1}{2}\sum_{\beta\in R_M^+}\beta$, is a Kähler–Einstein metric (up to a constant) on M compatible with J.*

The weight $\delta_{\mathfrak{m}}$ is called the *Koszul form*. If $M = G/K$ is defined by a set $\Pi_M = \{\alpha_{i_1},\dots,\alpha_{i_r}\}$, then we have $2\delta_{\mathfrak{m}} = c_{i_1}\Lambda_{i_1} + \cdots + c_{i_r}\Lambda_{i_r}$, where $\Lambda_{i_1},\dots,\Lambda_{i_r}$ are the fundamental weights corresponding to the elements of Π_M, and $c_{i_1},\dots,c_{i_r} \in \mathbb{Z}_+$.

Proposition 3.5. *Let J be the G-invariant complex structure on M induced from the invariant ordering R_M^+ given in Table 4. Then, the SO(7)-invariant Kähler–Einstein metric g_J corresponding to J is given (up to a scale) by $g = (3,2,5,4,7)$.*

Proof. First we compute the Koszul form $\delta_{\mathfrak{m}}$ corresponding to the invariant ordering R_M^+ given in Table 4. Since our flag manifold M is such that $\Pi_M = \{\alpha_1,\alpha_3\}$, it is $2\delta_{\mathfrak{m}} = c_1\Lambda_1 + c_3\Lambda_3$, where the positive numbers c_1 and c_3 are under investigation. Based on the relation $R_M^+ = R^+\backslash R_K^+$ we easily obtain $2\delta_{\mathfrak{m}} = 2\delta_G - 2\delta_K = 2(\Lambda_1+\Lambda_2+\Lambda_3) - \alpha_2$. Moreover, by using the relation $\alpha_i = \sum_{j=1}^{3} A_{ij}\Lambda_j$, where (A_{ij}) are the entries of the Cartan matrix, we have $\alpha_2 = -\Lambda_1 + 2\Lambda_2 - 2\Lambda_3$, thus $2\delta_{\mathfrak{m}} = 3\Lambda_1 + 4\Lambda_3$ and $\delta_{\mathfrak{m}} = 3/2\Lambda_1 + 2\Lambda_3$. According to Theorem 3.4, the Kähler–Einstein metric g_J which is compatible to the natural invariant complex structure J defined by the invariant ordering R_M^+, is given by $g_J = x_{\bar{\alpha}_1}\cdot(\cdot,\cdot)|_{\mathfrak{m}_1} + x_{\bar{\alpha}_3}\cdot$

$(\cdot,\cdot)|_{\mathfrak{m}_2} + x_{\bar\alpha_1+\bar\alpha_3}\cdot(\cdot,\cdot)|_{\mathfrak{m}_3} + x_{2\bar\alpha_3}\cdot(\cdot,\cdot)|_{\mathfrak{m}_4} + x_{\bar\alpha_1+2\bar\alpha_3}\cdot(\cdot,\cdot)|_{\mathfrak{m}_5}$, where the positive numbers x_{ξ_k} are given by $x_{\xi_k} = \varphi(\delta_{\mathfrak{m}},\alpha)$ (here the root $\alpha \in R_M^+$ is such that $\alpha \in \kappa^{-1}(\xi_k)$, and ξ_k is the associated $\mathfrak{t}$-root of $\mathfrak{m}_k$). By an easy computation we obtain the following values:

$$
\begin{aligned}
x_{\bar\alpha_1} &= \varphi(3/2\Lambda_1 + 2\Lambda_3, \alpha_1) = \frac{3\varphi(\Lambda_1,\alpha_1)}{2} = \frac{3\varphi(\alpha_1,\alpha_1)}{4} = \frac{3\varphi(\alpha_3,\alpha_3)}{2},\\
x_{\bar\alpha_3} &= \varphi(3/2\Lambda_1 + 2\Lambda_3, \alpha_3) = 2\varphi(\Lambda_3,\alpha_3) = \varphi(\alpha_3,\alpha_3),\\
x_{\bar\alpha_1+\bar\alpha_3} &= \varphi(3/2\Lambda_1 + 2\Lambda_3, \alpha_1+\alpha_2+\alpha_3) = \frac{3\varphi(\Lambda_1,\alpha_1)}{2} + 2\varphi(\Lambda_3,\alpha_3) = \frac{5\varphi(\alpha_3,\alpha_3)}{2},\\
x_{2\bar\alpha_3} &= \varphi(3/2\Lambda_1 + 2\Lambda_3, \alpha_2+2\alpha_3) = 4\varphi(\Lambda_3,\alpha_3) = 2\varphi(\alpha_3,\alpha_3),\\
x_{\bar\alpha_1+2\bar\alpha_3} &= \varphi(3/2\Lambda_1 + 2\Lambda_3, \alpha_1+2\alpha_2+2\alpha_3) = \frac{3\varphi(\Lambda_1,\alpha_1)}{2} + 4\varphi(\Lambda_3,\alpha_3) = \frac{7\varphi(\alpha_3,\alpha_3)}{2}.
\end{aligned}
$$

By substituting the value $\varphi(\alpha_3,\alpha_3) = 1$, and after a normalization, the result follows. □

Remark 3.6. Note that for the flag manifold $SO(7)/K$ defined by the PDD (III), for the invariant ordering R_M^+ given by in Table 4, we find that $\delta_{\mathfrak{m}} = 3/2\Lambda_2 + \Lambda_3$. The corresponding SO(7)-invariant Kähler–Einstein metric g_J, is given by (up to a scale) $g_J = (3,1,4,5,8)$.

Let us now consider the system

$$
r_1 - r_2 = 0, \qquad r_2 - r_3 = 0, \qquad r_3 - r_4 = 0, \qquad r_4 - r_5 = 0, \tag{21}
$$

where the components r_1, r_2, r_3, r_4 and r_5 are given by (20). In system (21) we substitute the dimensions $d_i = \dim \mathfrak{m}_i$ presented in Table 5 and the coefficients $x_1 = 3$, $x_2 = 2$, $x_3 = 5$, $x_4 = 4$ and $x_5 = 7$ of the Kähler–Einstein metric g_J. Then we obtain

Lemma 3.7. *The structure constants c_{12}^3, c_{14}^5, c_{22}^4 and c_{23}^5 of $M = SO(7)/U(1)\times U(2)$ with respect to the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$ are given by $c_{12}^3 = c_{14}^5 = c_{22}^4 = c_{23}^5 = 2/5$.*

By Lemma 3.7, Proposition 3.3 and Table 5, system (21) reduces to the following system (we also apply the normalization $x_1 = 1$):

$$
\left.
\begin{aligned}
&\frac{x_3x_4^2x_5 - x_2^3(x_4+2x_4x_5) - x_2^2x_3(-1+x_4^2-10x_4x_5+x_5^2) + x_2x_4(x_3^2-10x_3x_5+x_5(2+x_5))}{20x_2^2x_3x_4x_5} = 0,\\
&\frac{-10x_2^2x_5 - x_3x_4x_5 + 3x_2^3(1+x_5) + x_2(10x_3x_5 - 3x_3^2(1+x_5) + x_5(1+x_5))}{20x_2^2x_3x_5} = 0,\\
&\frac{-x_3x_4^2x_5 - 2x_2^3x_4(1+x_5) + 2x_2x_4(x_3^2-x_5)(1+x_5) + 2x_2^2(5x_4x_5 + x_3(1-x_4^2-4x_5+x_5^2))}{20x_2^2x_3x_4x_5} = 0,\\
&\frac{x_2^3x_4 + x_3x_4^2x_5 + x_2^2x_3(-1-10x_4+3x_4^2+8x_5-3x_5^2) + x_2x_4(x_3^2-x_5^2)}{20x_2^2x_3x_4x_5} = 0.
\end{aligned}
\right\} \tag{22}
$$

Any positive real solution $x_2 > 0$, $x_3 > 0$, $x_4 > 0$, $x_5 > 0$ of system (22), determines an SO(7)-invariant Einstein metric $(1, x_2, x_3, x_4, x_5) \in \mathbb{R}_+^5$ on M. One can obtain all these solutions by applying for example, the command `NSolve` in Mathematica.

Theorem 3.8. *The flag manifold $M = SO(7)/U(1)\times U(2) \cong SO(7)/U(1)^2 \times SU(2)$ admits (up to a scale) eight SO(7)-invariant Einstein metrics, which approximately are given as follows*

(a) $(1, 1.0231, 0.3089, 1.8751, 0.9999)$, (b) $(1, 1.0157, 0.2458, 0.5319, 0.9999)$,

(c) $(1, 0.5422, 0.9898, 0.5176, 0.6571)$, (d) $(1, 0.8251, 1.5063, 0.7877, 1.5217)$,

(e) $(1, 4/5, 1/5, 8/5, 3/5)$, (f) $(1, 4/3, 1/3, 8/3, 5/3)$,

(g) $(1, 2/3, 5/3, 4/3, 7/3)$, (h) $(1, 2/7, 5/7, 4/7, 3/7)$.

The metrics (e), (f), (g) and (h) are Kähler–Einstein.

Table 6
The values of H_g for the Einstein metrics which admits $M = \mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2) \cong \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$.

Invariant Einstein metrics	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Corresponding values of H_g	5.7677	5.6670	5.8968	5.8968	5.7748	5.7748	5.9232	5.9232

3.3. The isometry problem for the Einstein metrics

We will examine now the isometry problem for the homogeneous Einstein metrics stated in Theorem 3.8. We follow the method presented in [7].

Let $M = G/K$ be a generalized flag manifold with $\mathfrak{m} = \bigoplus_{i=1}^5 \mathfrak{m}_i$, $d_i = \dim \mathfrak{m}_i$, and $d = \sum_{i=1}^5 d_i = \dim M$. For any G-invariant Einstein metric $g = (x_1, x_2, x_3, x_4, x_5)$ on M we determine a normalized scale invariant given by $H_g = V_g^{1/d} S_g$, where S_g is the scalar curvature of g, and $V_g = \prod_{i=1}^5 x_i^{d_i}$ is the volume of the given metric g. Since, the scalar curvature is a homogeneous polynomial of degree -1 on the variables x_i (see Proposition 3.2), and the volume V_g is a monomial of degree d, the quantity $H_g = V_g^{1/d} S_g$ is a homogeneous polynomial of degree 0. Therefore, H_g is invariant under a common scaling of the variables x_i. If two metrics are isometric then they have the same scale invariant, so if the scale invariants H_g and $H_{g'}$ are different, then the metrics g and g' cannot be isometric. But if $H_g = H_{g'}$, we cannot immediately conclude if the metrics g and g' are isometric or not. For such a case we have to look at the group of automorphisms of G and check if there is an automorphism which permutes the isotropy summands and takes one metric to another. Kähler–Einstein metrics which correspond to equivalent invariant complex structures on M are isometric.

Consider our quotient $M = \mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2) \cong \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$, and let $g = (x_1, x_2, x_3, x_4, x_5)$ be an SO(7)-invariant Riemannian metric on M. By applying Proposition 3.2, we easily find that the scalar curvature S_g is given by

$$S_g = \sum_{k=1}^{5} \frac{d_k}{x_k} - \frac{c_{12}^3}{4}\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right) - \frac{c_{14}^5}{4}\left(\frac{x_1}{x_4x_5} + \frac{x_4}{x_1x_5} + \frac{x_5}{x_1x_4}\right)$$
$$- \frac{c_{23}^5}{4}\left(\frac{x_2}{x_3x_5} + \frac{x_3}{x_2x_5} + \frac{x_5}{x_2x_3}\right) - \frac{c_{22}^4}{4}\left(\frac{x_4}{x_2^2} + \frac{2}{x_4}\right).$$

Thus, by using Lemma 3.7 and the dimensions $d_i = \dim \mathfrak{m}_i$ of Table 5, we get that for a normalized SO(7)-invariant metric $g = (1, x_2, x_3, x_4, x_5)$, the scale invariant H_g is given by

$$H_g = \frac{-x_2^2 x_3 x_4 x_5^3}{10(x_2^4 x_3^2 x_4^2 x_5^4)^{\frac{15}{16}}}\big(x_3 x_4^2 x_5 + 2x_2^3 x_4(1+x_5) + 2x_2 x_4\big(-10x_3x_5 + x_3^2(1+x_5) + x_5(1+x_5)\big)$$
$$+ 2x_2^2\big(-5x_4x_5 + x_3\big(1 + x_4^2 - 4x_5 + x_5^2 - 10x_4(1+x_5)\big)\big)\big).$$

For the SO(7)-invariant Einstein metrics presented in Theorem 3.8 we obtain the following approximate values of the scale invariant H_g.

Relatively to Theorem 3.8, and in view of Table 6, we easily deduce that the Kähler–Einstein metrics (e) and (f) cannot be isometric with the Kähler–Einstein metrics (g) and (h). However, by [21, Theorem 5] we know $M = \mathrm{SO}(7)/\mathrm{U}(1) \times \mathrm{U}(2) \cong \mathrm{SO}(7)/\mathrm{U}(1)^2 \times \mathrm{SU}(2)$ admits two pairs of equivalent SO(7)-invariant complex structures (or equivalently, exactly two inequivalent complex structures), and thus, there are two pairs of isometric Kähler–Einstein metrics. Since $H_{(e)} = H_{(f)}$ and $H_{(g)} = H_{(h)}$, we have that

Corollary 3.9. *The Kähler–Einstein metrics (e) and (f), presented in Theorem 3.8, are isometric each other and the same is true for the pair (g) and (h), of the same theorem. Thus M admits precisely two (up to a scale) non-isometric SO(7)-invariant Kähler–Einstein metrics.*

Let now exam the isometry problem of the non-Kähler–Einstein metrics of M. The first two metrics (a) and (b) presented in Theorem 3.8 are non-isometric each other, since $H_{(a)} \neq H_{(b)}$, and for the same reason they also cannot be isometric with any of the metrics (c) and (d). In particular for the later Einstein metrics we obtain the same scale invariant $H_{(c)} = H_{(d)}$, thus we are not still able to conclude immediately if these metrics are isometric or not. However, we see that the metric (c), given by $(x_1 = 1,\ x_2 = 0.54221,\ x_3 = 0.98988,\ x_4 = 0.51767,\ x_5 = 0.65715)$ is obtained from the metric (d), i.e. the metric $(x_1 = 1,\ x_2 = 0.82510,\ x_3 = 1.50633,\ x_4 = 0.78776,\ x_5 = 1.52173)$, by dividing the components of the later metric with $1.52173 = x5$ and interchanging x_1 and x_5. Thus, it is sufficient to find a linear isomorphism $\sigma : \mathfrak{m}_1 \to \mathfrak{m}_5$, $\sigma : \mathfrak{m}_2 \to \mathfrak{m}_2$, $\sigma : \mathfrak{m}_3 \to \mathfrak{m}_3$ and $\sigma : \mathfrak{m}_4 \to \mathfrak{m}_4$ which induces an exchange of the variables x_1 and x_5 and remain fixed for the others. This isometry is induced by the action of the Weyl group $\mathcal{W} = \{s_\alpha\colon\ \alpha \in R\}$ of R. Indeed, recall that for any $\alpha, \beta \in R$ we have

$$s_\alpha(\beta) = \beta - \frac{2\varphi(\beta,\alpha)}{\varphi(\alpha,\alpha)}\alpha = \beta - (p-q)\alpha, \tag{23}$$

where the positive integers p, q are determined by the α-chain of roots through β: $-p\alpha+\beta,\dots,\beta,\dots,q\alpha+\beta$. By applying relation (23), one can easily see that the reflection $s_{\alpha_2+2\alpha_3}$ by the root $\alpha_2+2\alpha_3$ is such that

$$s_{\alpha_2+2\alpha_3}(\alpha_1)=\alpha_1+\alpha_2+2\alpha_3, \tag{24}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_2)=\alpha_2, \tag{25}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_3)=-(\alpha_2+\alpha_3), \tag{26}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_1+\alpha_2)=\alpha_1+2\alpha_2+2\alpha_3, \tag{27}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_2+\alpha_3)=-\alpha_3, \tag{28}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_1+\alpha_2+\alpha_3)=\alpha_1+\alpha_2+\alpha_3, \tag{29}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_2+2\alpha_3)=-(\alpha_2+2\alpha_3), \tag{30}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_1+\alpha_2+2\alpha_3)=\alpha_1, \tag{31}$$

$$s_{\alpha_2+2\alpha_3}(\alpha_1+2\alpha_2+\alpha_3)=\alpha_1+\alpha_2 \tag{32}$$

In view of the first column of Table 5, and due to relations (24)–(27) and (31)–(32), we conclude that the isometry $s_{\alpha_2+2\alpha_3}$ maps $\mathfrak{m}_1\to\mathfrak{m}_5$ and vice versa. Similarly, relations (26)–(28), (29), and (30), imply that the reflection $s_{\alpha_2+2\alpha_3}$ induces a linear map which maps the isotropy summands $\mathfrak{m}_2$, $\mathfrak{m}_3$ and $\mathfrak{m}_4$ onto themselves and leave them invariant. Thus the reflection $\sigma=s_{\alpha_2+2\alpha_3}$ shows that the metrics (c) and (d) are mutually isometric.

Corollary 3.10. *The non-Kähler–Einstein metrics* (c) *and* (d), *presented in Theorem 3.8, are isometric each other. Thus M admits precisely three* (*up to scale*) *non-isometric* SO(7)*-invariant Einstein metrics, which are not Kähler with respect to any* SO(7)*-invariant complex structure of M.*

Theorem A in introduction, is following now by Proposition 3.1, Theorem 3.8 and Corollaries 3.9 and 3.10.

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